Chapter 8

Advanced counting techniques

Chapter Summary

- Applications of Recurrence Relations
- Solving Linear Recurrence Relations
  - Homogeneous Recurrence Relations
  - Nonhomogeneous Recurrence Relations
- Divide-and-Conquer Algorithms and Recurrence Relations
- Generating Functions
- Inclusion-Exclusion
- Applications of Inclusion-Exclusion

§ 8.1 Applications of Recurrence Relations (1)

8.1.1 The concept of recurrence relations

Defination:

A sequence is a function from a subset of the set of integers (usually either the set {0,1,2,...} or the set {1,2,3,...}) to a set S.We use the notation a_n to denote the image of the integer n.We call a_n a term of the sequence.

We use the notation $\{a_n\}$ to describe the sequence.

§ 8.1 Applications of Recurrence Relations (2)

8.1.1 The concept of recurrence relations

Defination:

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence,namely,a_0,a_1,.., a_{n-1},for all integers n with $n \geq n_0$,where n_0 is a nonnegative integer.A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

§ 8.1 Applications of Recurrence Relations (3)

8.1.2 Modeling with recurrence relations

Example 1： Bacterial Reproduction

$$a_n = 2a_{n-1}, \quad a_0 = 5; \quad a_n = 2^n \times 5$$

Example 2： Compound Interest

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

$$P_n = (1.05)^n 1000$$

§8.1 Applications of Recurrence Relations (4)

8.1.2 Modeling with recurrence relations

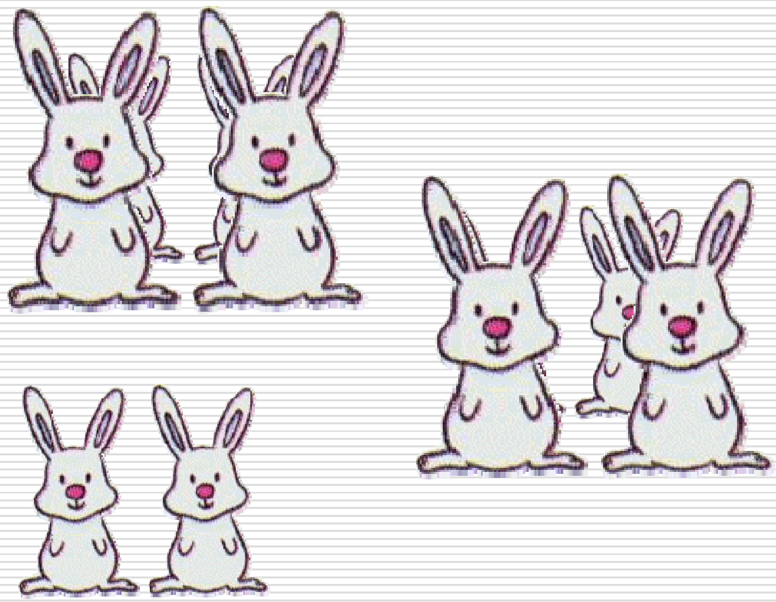
Example 3：Fibonacci Numbers

第1个月 1 对

第2个月 1 对

第3个月 2 对

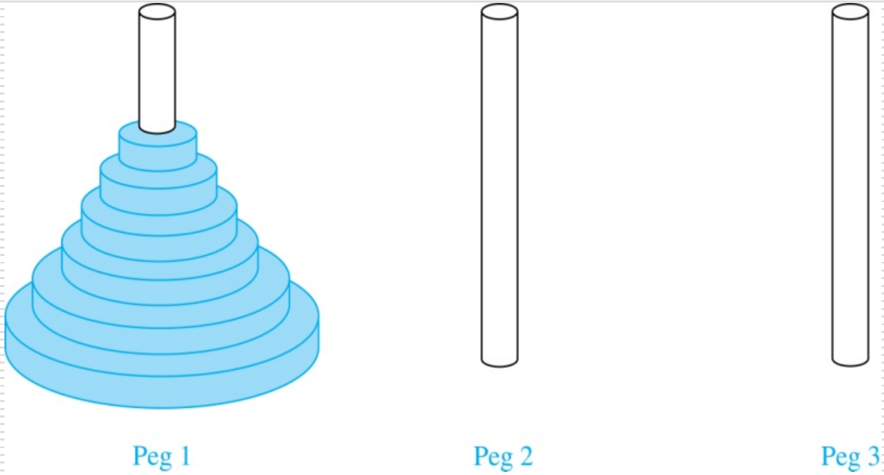
第4个月 3 对

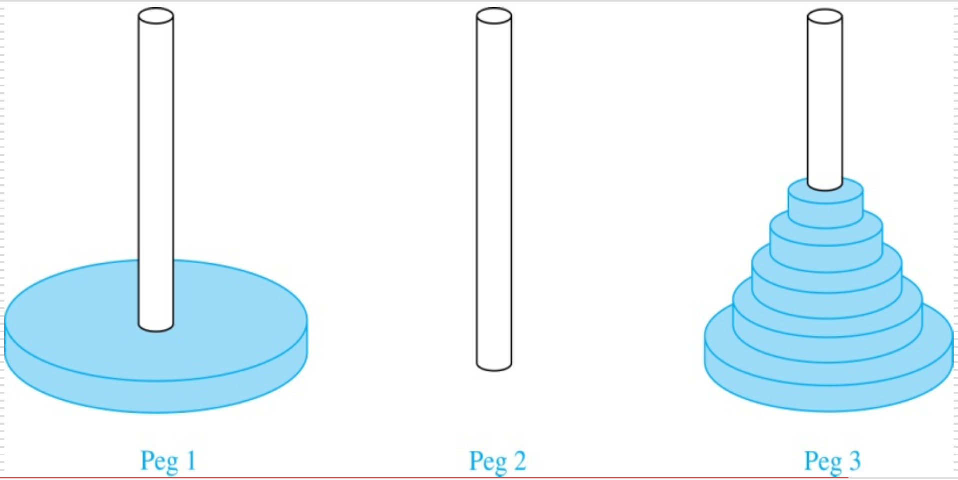


§ 8.1 Applications of Recurrence Relations (5)

8.1.2 Modeling with recurrence relations

Example 4: The Tower Hanoi





§8.1 Applications of Recurrence Relations (6)

8.1.2 Modeling with recurrence relations

Example 5: Bit Strings

Example 6: Codeword Enumeration

Example 7: Lancaster

§ 8.2 Solving Linear Recurrence Relations (1)

8.2.1 Linear homogeneous recurrence relation of degree k

Definition:

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers , and $c_k \neq 0$.

§ 8.2 Solving Linear Recurrence Relations (2)

8.2.1 Linear homogeneous recurrence relation of degree k

$F_n = F_{n-1} + F_{n-2}$ linear homogeneous recurrence relation of degree two

$a_n = a_{n-1} + a_{n-2}^2$ not linear

$H_n = 2H_{n-1} + 1$ not homogeneous

$B_n = nB_{n-1}$ coefficients are not constants

§ 8.2 Solving Linear Recurrence Relations (3)

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

Linear homogeneous recurrence relation with constant coefficients :

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0$$

Characteristic equation:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

（1） distinct root

Theorem 1： Let c_1 and c_2 be real numbers. Suppose that $r^2-c_1r-c_2=0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}$ if and only if $a_n=b_1r_1^n+b_2r_2^n$ for $n=0,1,2,\ldots$, where b_1 and b_2 are constants.

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

(1) distinct root

$$\begin{cases} a_0 = C_0 = b_1 + b_2 \\ a_1 = C_1 = b_1 r_1 + b_2 r_2 \end{cases}$$

$$b_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

$$b_2 = C_0 - b_1$$

$$= C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2}$$

§ 8.2 Solving Linear Recurrence Relations (6)

Example 1:求解下列递推关系的解。

$$\begin{cases} a_n = a_{n-1} + 2a_{n-2} \\ a_0 = 2, a_1 = 7 \end{cases}$$

Example 2:求解Fibonacci递推关系的解。

$$\begin{cases} F_n = F_{n-1} + F_{n-2} & n \geq 3 \\ F_1 = F_2 = 1 \end{cases}$$

§8.2 Solving Linear Recurrence Relations (7)

Example 3：

只有三个字母a,b,c组成的长度为n的一些单词将在通信信道上传输。必须满足下列条件：传输中不得有两个a连续出现在任一单词中。确定通信信道允许传输的单词个数。

§ 8.2 Solving Linear Recurrence Relations (8)

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

(1) distinct root

Theorem 2: Let c1,c2,…,ck be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if $a_n = b_1 r_1^n + b_2 r_2^n + \ldots + b_k r_k^n$, for n=0,1,2,…, where $b_1, b_2, \ldots, b_k$ are constants.

§ 8.2 Solving Linear Recurrence Relations (9)

Example: 求下列递推关系的解。

$$\begin{cases} a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} & (n \geq 3) \\ a_0 = 1, a_1 = 2, a_2 = 0 \end{cases}$$

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

（2） Multiple root

Theorem 3: Let c_1 and c_2 be real numbers wth $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = b_1 r_0^n + b_2 n r_0^n$ for $n = 0, 1, 2, \ldots$, where b_1 and b_2 are constants.

Example：求解下列递推关系的解。

$$\begin{cases} a_n = 2a_{n-1} - a_{n-2} \quad (n \geq 3) \\ a_1 = 2, a_2 = 3 \end{cases}$$

§ 8.2 Solving Linear Recurrence Relations (12)

Theorem 4: Let c1,c2,…,ck be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \dots - c_k = 0$ has t distinct roots $r_1, r_2, \dots, r_t$ with multiplicities $m_1, m_2, \dots, m_t$, respectively ,so that $m_i \geq 1$ for $i=1,2,\dots,t$ and $m_1+m_2+\dots+m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}+\dots+c_ka_{n-k}$ if and only if

$$\begin{aligned} a_n = {} & (b_{1,0}+b_{1,1}n+\dots+b_{1,m_1-1}n^{m_1-1})r_1^n \\ & +(b_{2,0}+b_{2,1}n+\dots+b_{2,m_2-1}n^{m_2-1})r_2^n \\ & +\dots+(b_{t,0}+b_{t,1}n+\dots+b_{t,m_t-1}n^{m_t-1})r_t^n \end{aligned}$$

for $n=0,1,2,3,\dots$, where $b_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i-1$.

Example：求解下述递推关系的解。

$$\begin{cases} a_n = -a_{n-1} + 3a_{n-2} + 5a_{n-3} + 2a_{n-4} \quad (n \geq 4) \\ a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 2 \end{cases}$$

Solving linear homogeneous recurrence relation with constant coefficients

（1）The characteristic equation

（2）The characteristic roots

（3）Distinct root？ Multiple root？

（4）The initial conditions

§8.2 Solving Linear Recurrence Relations (14)

Example：求解下述递推关系的解。

$$\begin{cases} a_n = 8a_{n-2} - 16a_{n-4} & (n \geq 4) \\ a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 2 \end{cases}$$

特征方程： $r^4 - 8r^2 + 16 = 0$

特征方程的根： $r_1 = 2$， $r_2 = -2$ 都是二重根

递推关系的解： $a_n = (b_1 + b_2 n) r_1^n + (b_3 + b_4 n) r_2^n$

8.2.3 Linear nonhomogeneous recurrence relations with constant coefficients

A **linear nonhomogeneous recurrence relation with constant coefficients** is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

where $c_1, c_2, \ldots c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n.

The recurrence relaton

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$, is called the **associated homogeneous recurrence relation .**

§ 8.2 Solving Linear Recurrence Relations (14)

Example:

$$a_n = a_{n-1} + 2^n$$

$$a_n = a_{n-1}$$

$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$$

$$a_n = a_{n-1} + a_{n-2}$$

$$a_n = 3a_{n-1} + n3^n$$

$$a_n = 3a_{n-1}$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

8.2.3 Linear nonhomogeneous recurrence relations with constant coefficients

Theorem 1:

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence ralation with constant coefficients

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}+F(n),$$

then every solutionis of the form $\{a_n^{(p)}+a_n^{(h)}\}$,where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence ralation

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}.$$

§ 8.2 Solving Linear Recurrence Relations (16)

$$a_n^{(h)} = c_1 a_{n-1}^{(h)} + c_2 a_{n-2}^{(h)} + \cdots + c_k a_{n-k}^{(h)}$$

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n)$$

$$a_n^{(h)} + a_n^{(p)} = c_1(a_{n-1}^{(h)} + a_{n-1}^{(p)}) + c_2(a_{n-2}^{(h)} + a_{n-2}^{(p)}) + \cdots + c_k(a_{n-k}^{(h)} + a_{n-k}^{(p)}) + F(n)$$

$$a_n = a_n^{(p)} + a_n^{(h)} = a_n^{(p)} + \sum_{i=1}^{k} b_i r_i^n$$

§8.2 Solving Linear Recurrence Relations (17)

初始条件：$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}$

$$b_1 + b_2 + \cdots + b_k = C_0{}^*$$

$$r_1 b_1 + r_2 b_2 + \cdots + r_k b_k = C_1{}^*$$

$$\cdots\cdots$$

$$r_1^{k\ 1} b_1 + r_2^{k\ 1} b_2 + \cdots + r_k^{k\ 1} b_k = C_{k\ 1}{}^*$$

其中：$C_i^* = C_i - a_i^{(p)}$

求解非齐次线性常系数递推关系

（1）求出相伴齐次递推关系的解的形式

（2）根据非齐次递推关系F(n)的不同，确定特解的形式

（3）将特解形式代入到递推关系中求出待定系数，确定特解

（4）给出递推关系解的形式，
即 特解 + 相伴齐次递推关系解的形式

（5）由初始条件求出相伴齐次递推关系的系数，确定递推关系的解

§8.2 Solving Linear Recurrence Relations (18)

（1）当F(n)是 n 的 k 次多项式时，可设递推关系的特解形式为：

$$a^{(p)} = A_0 n^k + A_1 n^{k-1} + \cdots + A_k$$

式中A_0,A_1,…,A_k为待定常数。

Example 1：求解下列递推关系

$$\begin{cases} a_n = -2a_{n-1} + n + 3 \\ a_0 = 3 \end{cases}$$

§8.2 Solving Linear Recurrence Relations (19)

当相伴齐次递推关系的特征根为 1 的 m （m ≥ 1）重根时，特解的形式为：

$$a^{(p)} = (A_0 n^k + A_1 n^{k-1} + \cdots + A_k)\ n^m$$

式中 $A_0, A_1, \ldots, A_k$ 为待定常数。

Example 2: 求解下列递推关系

$$\begin{cases} a_n = a_{n-1} + 2(n-1) \quad (n \geq 2) \\ a_1 = 2 \end{cases}$$

§8.2 Solving Linear Recurrence Relations (20)

（2）当F(n)是β^n的形式时，又可分为如下两种情形：I、如果β不是相伴递推关系式的特征根时，可设特解的形式为： $a_n^{(p)} = A\beta^n$

II、如果β是相伴递推关系式的k重特征根时（k≥1），可设特解的形式为：$a_n^{(p)} = n^k A\beta^n$

（3）当F(n)是 $P_m(n)\beta^n$ 的形式时

$$a_n^{(p)} = n^k (A_0 n^m + A_1 n^{m-1} + \cdots + A_m)\beta^n$$

§ 8.2 Solving Linear Recurrence Relations (21)

Theorem 2:

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n),$$

where $c_1, c_2, \ldots c_k$ are real numbers and

$$F(n) = (b_tn^t + b_{t-1}n^{t-1} + \ldots + b_1n + b_0)\beta^n,$$

where $b_0, b_1, \ldots, b_t$ and β are real numbers.

When β is not a root of the characteristic equation of the associated linear homogeneous relation, there is a particular solution of the form

$$(p_tn^t + p_{t-1}n^{t-1} + \ldots + p_1n + p_0)\ \beta^n.$$

When β is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m(p_tn^t + p_{t-1}n^{t-1} + \ldots + p_1n + p_0)\ \beta^n.$$

Example

假设线性非齐次递推关系为:

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

当 $F(n) = 3^n$ $F(n) = n3^n$ $F(n) = n^2 2^n$

$F(n) = (n^2 + 1)3^n$

时，上述线性非齐次递推关系的特解是什么?

§ 8.3 Divide-and-Conquer Algorithms and Recurrence relations (1)

Suppose that a recursive algorithm divides a problem of size n into t subproblems,where each subproblem is of size n/b.Also suppose that a total of g(n) extra operations are required in the conquer step of the algorithm to combine the solutions of the subproblems into a solution of the original problem.Then ,if f(n) represents the number of operations required to solve the problem of size n,it follows that f satisfies the recurrence relation

$$f(n) = t\, f(n/b) + g(n).$$

This is called a divide-and-conquer recurrence relation.

§ 8.3 Divide-and-Conquer Algorithms and Recurrence relations (2)

Theorem 1:

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a\, f(n/b) + c$$

whenever n is divisible by b, where $a \ge 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}), & \text{if } a > 1 \\ O(\log n), & \text{if } a = 1 \end{cases}$$

§8.3 Divide-and-Conquer Algorithms and Recurrence relations (3)

O符号定义：

假设 f 和 g 是从整数集合或实数集合到实数集合的函数，如果有常数 C 和 k，使得只要 x > k，就有 $|f(x)| \le C|g(x)|$，则f (x)是O(g (x))的。

Furthermore, when $n = b^k$, where k is a positive integer, **$f(n) = C_1 n^{\log_b a} + C_2$**,

where $C_1 = f(1) + c/(a-1)$ and

$C_2 = -c/(a-1)$.

§ 8.3 Divide-and-Conquer Algorithms and Recurrence relations (4)

Theorem2 (Master Theorem):

Let f be increasing function that satisfies the recurrence relation $f(n) = af(n/b) + cn^d$, whenever $n = b^k$, where k is a positive integer, $a \ge 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n)\ is \begin{cases} O(n^d) & if\ \ a < b^d \\ O(n^d \log n) & if\ \ a = b^d \\ O(n^{\log_b a}) & if\ \ a > b^d \end{cases}$$

§ 8.4 Generating Functions (1)

8.4.1 The concept of generating functions

（1）Ordinary generating function

Definition:

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

§8.4 Generating Functions (2)

Example 1：给出序列1，1，1，1，1的普通生成函数

$$f(x) = 1 + x + x^2 + x^3 + x^4 = \sum_{k=0}^{4} x^k = \frac{x^5 - 1}{x - 1}$$

Example 2：给出序列 $\left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n}\right)$ 的普通生成函数

$$f(x) = \binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

§8.4 Generating Functions (3)

Example 3：给出序列 $\left(\binom{n-1}{0},-\binom{n}{1},\binom{n+1}{2},\cdots,(-1)^k\binom{n+k-1}{k},\cdots\right)$ 的普通生成函数

$$f(x)=\binom{n-1}{0}-\binom{n}{1}x^1+\binom{n+1}{2}x^2+\cdots+(-1)^k\binom{n+k-1}{k}x^k+\cdots$$

$$=\sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k=(1+x)^{-n}$$

8.4.1 The concept of generating functions

（2）Exponential generating function

Definition:

The exponential generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$f_e(x) = a_0 + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + \cdots + a_n \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

§8.4 Generating Functions (5)

Example 4：给出序列 $(P(n,0),P(n,1),\ldots,P(n,n))$ 的指数生成函数

$$f_e(x)=P(n,0)+P(n,1)\frac{x^1}{1!}+P(n,2)\frac{x^2}{2!}+\cdots+P(n,n)\frac{x^n}{n!}+0+\cdots$$

$$=\binom{n}{0}+\binom{n}{1}x^1+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n=(1+x)^n$$

§8.4 Generating Functions (6)

Example 5：给出序列 $(1, a, a^2, \cdots, a^n, \cdots)$ 的指数生成函数

$$f_e(x) = a_0 + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + \cdots + a_n \frac{x^n}{n!} + \cdots$$

$$= 1 + a \frac{x^1}{1!} + a^2 \frac{x^2}{2!} + \cdots + a^n \frac{x^n}{n!} + \cdots$$

$$= \sum_{k=0}^{\infty} a^k \frac{x^k}{k!} = e^{ax}$$

§ 8.4 Generating Functions (1)

8.4.2 Useful facts about power series

（1） The foundational operation of generating function

Theorem 1： $Let\ f(x)=\sum_{k=0}^{\infty} a_k x^k\ and\ g(x)=\sum_{k=0}^{\infty} b_k x^k .Then$

$$f(x)+g(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k\ and$$

$$f(x)g(x)=\sum_{k=0}^{\infty}(\sum_{j=0}^{k} a_j b_{k-j})x^k$$

8.4.2 Useful facts about power series

(2) The extended binomial theorem

Definition:

Let α be a real number and let k be a nonnegative integer. Then the extended binomial coefficient $C(\alpha,k)$ is defined by

$$\binom{\alpha}{k} = \begin{cases} \alpha(\alpha-1)\cdots(\alpha-k+1)/k! & \text{if } k>0 \\ 1 & \text{if } k=0 \end{cases}$$

8.4.2 Useful facts about power series

（2） The extended binomial theorem

Example 1

$$\binom{-2}{3} = (-2)(-2-1)(-2-2)/3! = (-2)(-3)(-4)/3! = -4$$

$$\binom{1/2}{3} = (1/2)(1/2-1)(1/2-2)/3! = (1/2)(-1/2)(-3/2)/6 = 1/16$$

8.4.2 Useful facts about power series

(2) The extended binomial theorem

The extended binomial theorem:

Let x be a real number with |x| < 1 and let α be a real number. Then

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{k}$$

§ 8.4 Generating Functions (5)

8.4.2 Useful facts about power series

（2） The extended binomial theorem

Example 2 对于|x|<1 的任意 x，证明：

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k$$

当n=1时 $(1+x)^{-1} = \frac{1}{(1+x)} = \sum_{k=0}^{\infty} (-1)^k x^k$

用-x替代x $(1\text{-}x)^{-1} = \frac{1}{(1\text{-}x)} = \sum_{k=0}^{\infty} x^k$

§8.4 Generating Functions (6)

8.4.2 Useful facts about power series

Example 3 证明 $(1-4x)^{-1/2}$ 是下列序列的普通生成函数。

$$\left(\binom{0}{0},\binom{2}{1},\binom{4}{2},\cdots,\binom{2n}{n},\cdots\right)$$

§8.4 Generating Functions (1)

8.4.3 Counting problems and generating functions

(1) 从 n 个不同物体中选取 r 个物体，其方法数为 $(1+x)^n$ 的幂级数展开式中 x^r 的系数 C(n,r) 。

(2) 从 n 个不同物体中可重复选取 r 个物体，其方法数为 $(1+x+x^2+x^3+\ldots)^n$ 的幂级数展开式中 x^r 的系数。

§8.4 Generating Functions (2)

8.4.3 Counting problems and generating functions

例： 从 n 个不同物体中可重复选取 r 个物体的方法数为： $\binom{n+r-1}{r}$

$$f(x) = (1 + x + x^2 + \cdots)^n = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

8.4.3 Counting problems and generating functions

例： 从 n 个不同物体中可重复选取 r 个物体，每个物体至少选一次的方法数为：$\binom{r-1}{n-1}$

$$f(x) = (x + x^2 + \cdots)^n = \sum_{r=0}^{\infty} \binom{r-1}{n-1} x^r$$

§ 8.4 Generating Functions (4)

8.4.3 Counting problems and generating functions

例：从 n 个不同物体中选取 r 个物体的方法数为：$\binom{n}{r}$

$$f(x)=(1+x)^n=\sum_{r=0}^{\infty}\binom{n}{r}x^r=\sum_{r=0}^{\infty}P(n,r)\frac{x^r}{r!}$$

例： 求 1,3,5,7,9 五个数字组成 r 位数的个数。其中7,9 出现的次数为偶数，其它数字出现的次数不加限制。

8.4.4 Using generating functions to solve recurrence relations

Example 1:用生成函数求解下列递推关系式

$$\begin{cases} a_n = 3a_{n-1} & (n \geq 1) \\ a_0 = 2 \end{cases}$$

特征根 r=3

所以递推关系的解形式为：$a_n=b3^n$

由初始条件解为： $a_n=2\times 3^n$

§8.4 Generating Functions (2)

利用生成函数求解递推关系的步骤:

（1）用 f(x)表示序列（$a_0,a_1,a_2,\ldots$）的普通生成函数。即

$$f(x)=\sum_{n=0}^{\infty}a_n x^n$$

（2）利用递推关系将生成函数化为关于 f(x)的方程。

（3）求解出 f(x)。

（4）将f(x)表达式展开成幂级数形式，x^n 的系数就是递推关系的解。

8.4.4 Using generating functions to solve recurrence relations

Example 2:用生成函数求解下列递推关系

$$\begin{cases} a_n = a_{n-1} + 2(n-1)\ (n \geq 2) \\ a_1 = 2 \end{cases}$$

8.4.4 Using generating functions to solve recurrence relations

Example 3:用生成函数求解下列递推关系

$$\begin{cases} a_n = na_{n-1} + (-1)^n & n \geq 2 \\ a_0 = 1, a_1 = 0 \end{cases}$$

$$\begin{cases} a_n = na_{n-1} + (-1)^n & n \geq 2 \\ a_0 = 1, a_1 = 0 \end{cases}$$

设 $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ 下面序列的指数生成函数

$$(a_0, a_1, a_2, \cdots, a_n, \cdots)$$

$$f(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} [na_{n-1} + (-1)^n] \frac{x^n}{n!} = 1 + \sum_{n=2}^{\infty} na_{n-1} \frac{x^n}{n!} + \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$= 1 + x \sum_{n=2}^{\infty} a_{n-1} \frac{x^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n!} + (1-x) - (1-x)$$

$$= 1 + x \sum_{n=1}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} - (1-x)$$

$$= 1 + x \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} - x + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} - (1-x)$$

$$= x \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = xf(x) + e^{-x}$$

所以 $f(x) = \frac{e^{-x}}{1-x}$

$$f(x) = \frac{e^{-x}}{1-x} = \frac{1}{1-x} e^{-x} = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} n! \left[\sum_{k=0}^{n} (-1)^k \frac{1}{k!}\right] \frac{x^n}{n!}$$

8.4.5 Proving identities via generating functions

Example 1:设 n 为任意正整数，证明:

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

§ 8.5 Inclusion-exclusion (1)

8.5.1 The principle of inclusion-exclusion

The principle of inclusion-exclusion:

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n|$$

$$= \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

§8.5 Inclusion-exclusion (2)

8.5.1 The principle of inclusion-exclusion

Example 1：某班有学生60名，其中24名学生选修 A 课程，28名学生选修 B 课程，26名学生选修 C 课程，10名学生既选修 A 又选修 B，8名学生既选修 A 又选修 C，14名学生既选修 B 又选修 C，6名学生3门课程都选修了。问有多少学生对这3门课程都没有选修？

§ 8.6 Applications of inclusion-exclusion

（1） An alternative form of inclusion-exclusion

$$N(P_1', P_2', \ldots, P_n') = |S| - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

$$= N - \sum_{i=1}^{n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i, P_j) - \sum_{1 \le i < j < k \le n} N(P_i, P_j, P_k)$$

$$+ \cdots + (-1)^n N(P_1, P_2, \cdots P_n)$$

§8.6 Applications of inclusion-exclusion

(2) Counting the number of primes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36				

(3) The number of onto functions

§ 8.6 Applications of inclusion-exclusion

Theorem:

Let m and n be positive integers with m $\geq$ n.Then, there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1} C(n,n-1)1^m$$

onto functions from a set with m elements to a set with n elements.

§8.6 Applications of inclusion-exclusion

（4） Derangements

Theorem:

The number of derangements of a set with n elements is

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

§ 8.6 Applications of inclusion-exclusion

（4） Derangements

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} + \cdots$$

$$e^{-1} = \frac{D_n}{n!} + (-1)^{n+1} \frac{1}{(n+1)!} + (-1)^{n+2} \frac{1}{(n+2)!} \cdots$$

$$\left| e^{-1} - \frac{D_n}{n!} \right| < \frac{1}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{D_n}{n!} = e^{-1}$$

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

$$D_1 = 0, D_2 = 1$$