2011-2012 学年第二学期高等数学试题 (A) 参考答案

一、填空题（每小题4 分，共 16 分）

1．$\underline{\frac{1}{2}(1-\mathrm{e}^{-4})}$，2．$\underline{2}$，3．$\underline{i-\frac{1}{2}j+\frac{3}{2}k}$ ，4．$\underline{\frac{2}{3}\pi R^3}$

二、选择题（每小题4 分，共 16 分）1 (A) 2 (A) 3 (B) 4 (D)

三、（16 分）

1. 解：先拆项 $f(x)=\frac{1}{x-1}+\frac{1}{x+2}$。分别将$\frac{1}{x-1}$与$\frac{1}{x+2}$展开成$(x-2)$的幂级数。

$$\frac{1}{x-1}=\frac{1}{1+(x-2)}=\sum_{n=0}^{\infty}(-1)^n(x-2)^n,|x-2|<1;$$

$$\frac{1}{x+2}=\frac{1}{4+(x-2)}=\frac{1}{4}\frac{1}{1+\frac{x-2}{4}}=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(\frac{x-2}{4})^n,\left|\frac{x-2}{4}\right|<1.$$

$$f(x)=\sum_{n=0}^{\infty}(-1)^n(1+\frac{1}{4^{n+1}})(x-2)^n,|x-2|<1.$$

2. 解：Σ为$z=4-2x-\frac{4}{3}y$

$$\iint_{\Sigma}(z+2x+\frac{4}{3}y)\mathrm{d}S=\iint_{D_{xy}}4\times\frac{\sqrt{61}}{3}\mathrm{d}x\mathrm{d}y=4\times\frac{\sqrt{61}}{3}\iint_{D_{xy}}\mathrm{d}x\mathrm{d}y=4\sqrt{61}$$

四、(14 分)

1．求直线$L:\begin{cases}2x-y+z-1=0\\ x+y-z+1=0\end{cases}$ 在平面 $\pi:x+2y-z=0$ 上的投影直线方程。

解：过直线L的平面束方程为$\lambda(2x-y+z-1)+\mu(x+y-z+1)=0,$

即 $(2\lambda+\mu)x+(-\lambda+\mu)y+(\lambda-\mu)z+(-\lambda+\mu)=0,$ ①

则与平面π垂直的平面π_1的法向向量为$\vec{n}_1=\{2\lambda+\mu,-\lambda+\mu,\lambda-\mu\}.$

由题意知$\vec{n}_1\perp\vec{n}$，其中$\vec{n}=\{1,2,-1\}$，从而$(2\lambda+\mu)+2(-\lambda+\mu)-(\lambda-\mu)=0$，

解得$\lambda=4\mu$，带回①得与平面π垂直的平面方程为$3x-y+z-1=0$。

所求直线L在平面π上的投影直线应为平面π与平面π_1的交线，即

$$\begin{cases}x+2y-z=0\\ 3x-y+z-1=0\end{cases}$$

2. 解：Ω的上、下界面分别为$z=xy$与$z=0$。

$$I=\iint_{D_{xy}}\mathrm{d}\sigma\int_0^{xy}xy^2z^3\mathrm{d}z=\frac{1}{4}\iint_{D_{xy}}x^5y^6\mathrm{d}\sigma=\frac{1}{4}\int_0^1\mathrm{d}x\int_0^x x^5y^6\mathrm{d}y=\frac{1}{28}\int_0^1 x^{12}\mathrm{d}x=\frac{1}{364}$$

五、（12 分）

1. 解： 设$F(x)=\int_0^x f(t)\mathrm{d}t$，则$F'(x)=f(x)$，故

$$\int_0^1\mathrm{d}x\int_x^1\mathrm{d}y\int_x^y f(x)f(y)f(z)\mathrm{d}z=\int_0^1 f(x)\mathrm{d}x\int_x^1 f(y)\mathrm{d}y\int_x^y f(z)\mathrm{d}z$$
$$=\int_0^1 f(x)\mathrm{d}x\int_x^1 f(y)F(z)\Big|_x^y\mathrm{d}y=\int_0^1 f(x)\mathrm{d}x\int_x^1(F(y)-F(x))\mathrm{d}F(y)$$
$$=\int_0^1 f(x)[\frac{1}{2}F^2(y)-F(x)F(y)]\Big|_x^1\mathrm{d}x$$
$$=\int_0^1 f(x)[\frac{1}{2}F^2(1)-F(x)F(1)-(\frac{1}{2}F^2(x)-F^2(x))]\mathrm{d}x$$
$$=\int_0^1[\frac{1}{2}F^2(1)-F(x)F(1)+\frac{1}{2}F^2(x)]\mathrm{d}F(x)$$
$$=(\frac{1}{2}F^2(1)F(x)-\frac{1}{2}F^2(x)F(1)+\frac{1}{6}F^3(x))\Big|_0^1$$
$$=(\frac{1}{2}F^3(1)-\frac{1}{2}F^3(1)+\frac{1}{6}F^3(1))-(\frac{1}{2}F^2(1)F(0)-\frac{1}{2}F^2(0)F(1)+\frac{1}{6}F^3(0))$$
$$=\frac{1}{6}F^3(1)-(\frac{1}{2}F^2(1)F(0)-\frac{1}{2}F^2(0)F(1)+\frac{1}{6}F^3(0))$$

由于$F(x)=\int_0^x f(t)\mathrm{d}t$， $F(1)=\int_0^1 f(t)\mathrm{d}t$， $F(0)=\int_0^0 f(t)\mathrm{d}t=0$，

故 $\int_0^1\mathrm{d}x\int_x^1\mathrm{d}y\int_x^y f(x)f(y)f(z)\mathrm{d}z=\frac{1}{6}(\int_0^1 f(x)\mathrm{d}x)^3=\frac{1}{6}A^3$。

2. 解：利用高斯公式

$$I=3\iiint_{\Omega}(x^2+y^2+z^2)\mathrm{d}V=3\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^1 r^4\mathrm{d}r=\frac{12}{5}\pi$$ 。

六、（10 分）

解：利用格林公式，记 $J=\int_L P\mathrm{d}x+Q\mathrm{d}y$，

$P(x,y)=3x^2y,Q(x,y)=x^3+x-2y,$ 并且$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=3x^2+1-3x^2=1$，

由于曲线L不封闭，故添加辅助线L_1；沿y轴由点$B(0,2)\to O(0,0)$

则$\int_{L_1}P\mathrm{d}x+Q\mathrm{d}y=\int_{L_1}Q(0,y)\mathrm{d}y=\int_2^0(-2y)\mathrm{d}y=\int_0^2 2y\mathrm{d}y=4,$

然后在L_1与L围成的区域D上用格林公式（边界取正向），则：

$$\int_{L_1+L} P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}\sigma = \iint_D 1\,\mathrm{d}\sigma = \frac{2^2\pi}{4} - \frac{1^2\pi}{2} = \frac{\pi}{2}\text{。}$$

故 $J = \int_L P\mathrm{d}x + Q\mathrm{d}y = \int_{L_1+L} P\mathrm{d}x + Q\mathrm{d}y - \int_{L_1} P\mathrm{d}x + Q\mathrm{d}y = \frac{\pi}{2} - 4$

七、（10 分）

解： $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = -2, \frac{\partial v}{\partial x} = 1, \frac{\partial v}{\partial y} = 3$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\ ,$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -2\frac{\partial z}{\partial u} + 3\frac{\partial z}{\partial v},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= -2\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + 3\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(-2\frac{\partial z}{\partial u} + 3\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(-2\frac{\partial z}{\partial u} + 3\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(-2\frac{\partial z}{\partial u} + 3\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= 4\frac{\partial^2 z}{\partial u^2} - 12\frac{\partial^2 z}{\partial u \partial v} + 9\frac{\partial^2 z}{\partial v^2}$$

$$6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 25\frac{\partial^2 z}{\partial u \partial v},\quad 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 5\frac{\partial z}{\partial v}\text{。}$$

由此得出：$5\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial v}$

八、（6 分）

解：为椭球面 S 上点 $P(x,y,z)$ 处的法向量是 $\vec{n} = \{2x, 2y - z, 2z - y\}$，

点 P 处的切平面与 xOy 面垂直的充要条件是 $\vec{n} \cdot \vec{k} = 0$(取$\vec{k}$=$\{0,0,1\}$)

即 $2z - y = 0$。所以点 P 的轨迹 C 的方程为 $\begin{cases} 2z - y = 0, \\ x^2 + y^2 + z^2 - yz = 1, \end{cases}$

即
$$\begin{cases} 2z - y = 0, \\ x^2 + \dfrac{3}{4}y^2 = 1. \end{cases}$$

上述曲线在 xOy 平面上的投影曲线为 $\begin{cases} z = 0, \\ x^2 + \dfrac{3}{4}y^2 = 1. \end{cases}$

取 $D = \{(x, y) \mid x^2 + \dfrac{3}{4}y^2 \le 1\}$，记 Σ 的方程为 $z = z(x, y), (x, y) \in D$，由于

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \left(\frac{2x}{y - 2z}\right)^2 + \left(\frac{2y - z}{y - 2z}\right)^2} = \frac{\sqrt{4 + y^2 + z^2 - 4yz}}{|y - 2z|},$$

所以

$$I = \iint_D \frac{(x + \sqrt{3})|y - 2z|}{\sqrt{4 + y^2 + z^2 - 4yz}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \mathrm{d}x\mathrm{d}y = \iint_D (x + \sqrt{3})\mathrm{d}x\mathrm{d}y = \sqrt{3}\iint_D \mathrm{d}x\mathrm{d}y = 2\pi$$