2010-2011 学年第二学期高等数学试题 (A) 参考答案

一、填空题（每小题 4 分，共 20 分）

（1） 0 (2)$\dfrac{x-2}{-2}=\dfrac{y-4}{3}=\dfrac{z}{1}$ (2)$\dfrac{\vec{i}+\vec{j}+\vec{k}}{3}$ (4) 36π (5)$\dfrac{4}{9}$

二、选择题（每小题 4 分，共 20 分）

1 (C) 2 (B) 3 (D) 4 (A) 5(B)

三、解答题（1~6 题每题 8 分，第 7 题 12 分，共 60 分）

1. 解：因为$\dfrac{\partial g}{\partial x}=y\dfrac{\partial f}{\partial u}+x\dfrac{\partial f}{\partial v},\dfrac{\partial g}{\partial y}=x\dfrac{\partial f}{\partial u}-y\dfrac{\partial f}{\partial v}$

所以$\dfrac{\partial^2 g}{\partial x^2}=y^2\dfrac{\partial^2 f}{\partial u^2}+2xy\dfrac{\partial^2 f}{\partial u\partial v}+x^2\dfrac{\partial^2 f}{\partial v^2}+\dfrac{\partial f}{\partial v}$

$$\frac{\partial^2 g}{\partial y^2}=x^2\frac{\partial^2 f}{\partial u^2}-2xy\frac{\partial^2 f}{\partial u\partial v}+y^2\frac{\partial^2 f}{\partial v^2}-\frac{\partial f}{\partial v}$$

故$\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial g^2}=(x^2+y^2)\dfrac{\partial^2 f}{\partial u^2}+(x^2+y^2)\dfrac{\partial^2 f}{\partial v^2}=x^2+y^2$

2解$:S_1=\{1,-7,-5\},S_2=\{-2,4,-6\}$过$l_1$平面方程为$:\lambda(2x+y-z-1)+\mu$

$(3x-y+2z-2)=0$即$(2\lambda+3\mu)x+(\lambda-\mu)y+(-\lambda+2\mu)z-\lambda-2\mu=0$

由$\vec{n}\cdot\overrightarrow{S_2}=0$得$-2(2\lambda+3\mu)+(\lambda-\mu)+(-\lambda+2\mu)(-6)=0$得$\dfrac{\mu}{\lambda}=\dfrac{3}{11}$

$\therefore$平面方程为$31x+8y-5z-17=0$

3.解：连接BA，在$L+\overline{BA}$内做正向小圆周C_R:x^2+y^2=R^2，

取R>0适当小，使C_R与$L+\overline{BA}$不相交，且取逆时针方向为正向，

在以$L+\overline{BA}$+$C_R{}^-$为边界的复连域D内，

$P(x.y)=\dfrac{x+y}{x^2+y^2},Q(x.y)=\dfrac{x-y}{x^2+y^2},\dfrac{\partial P}{\partial y}=\dfrac{\partial P}{\partial y}$,由格林公式有

$\displaystyle\int_{L+\overline{BA}+C_R^-}\frac{(x+y)dx-(x-y)dy}{x^2+y^2}$=0,故

$$\int_{L}\frac{(x+y)dx-(x-y)dy}{x^2+y^2}=\int_{\overline{AB}}\frac{(x+y)dx-(x-y)dy}{x^2+y^2}+\int_{C_R}\frac{(x+y)dx-(x-y)dy}{x^2+y^2}$$

$$\int_{\overline{AB}}\frac{(x+y)dx-(x-y)dy}{x^2+y^2}\overset{y=-\pi}{=}\int_{\pi}^{-\pi}\frac{x-\pi}{x^2+\pi^2}dx=0+\int_{\pi}^{-\pi}\frac{-\pi}{x^2+\pi^2}dx=\frac{\pi}{2},$$

$$\int_{C_R}\frac{(x+y)dx-(x-y)dy}{x^2+y^2}\overset{\begin{cases}x=R\cos t\\ y=R\sin t\end{cases}}{=}$$

$$\int_{0}^{2\pi}\frac{(R\cos t+R\sin t)(-R\sin t)-(R\cos t-R\sin t)(R\cos t)}{R^2}dt=-2\pi,$$

$$\therefore\int_{L}\frac{(x+y)dx-(x-y)dy}{x^2+y^2}=-\frac{3}{2}\pi$$

4. 格林公式的叙述和证明见课本，此处略去。

解: $\int_{\widehat{ABO}}(\mathrm{e}^x\sin y-my)\mathrm{d}x+(\mathrm{e}^x\cos y-m)\mathrm{d}y$

$=\oint_{L}(\mathrm{e}^x\sin y-my)\mathrm{d}x+(\mathrm{e}^x\cos y-m)\mathrm{d}y+\int_{\overline{AO}}(\mathrm{e}^x\sin y-my)\mathrm{d}x+(\mathrm{e}^x\cos y-m)\mathrm{d}y$

$=\iint\limits_{D}\left[\mathrm{e}^x\cos y-\mathrm{e}^x\cos y+m\right]\mathrm{d}\sigma+\int_{\overline{AO}}(\mathrm{e}^x\sin y-my)\mathrm{d}x+(\mathrm{e}^x\cos y-m)\mathrm{d}y$

$=m\iint\limits_{D}\mathrm{d}\sigma+\int_{a}^{0}0\mathrm{d}x=\dfrac{m\pi}{8}a^2$

5解: 因为: $\lim\limits_{n\to\infty}\left|\dfrac{(2n+4)x^{2n+3}}{(n+1)!}\cdot\dfrac{n!}{(2n+2)x^{2n+1}}\right|=\lim\limits_{n\to\infty}\dfrac{(2n+4)x^2}{(n+1)(2n+2)}=0<1,$

故该级数的收敛域为$(-\infty,+\infty)$.

设$S(x)=\sum\limits_{n=1}^{\infty}\dfrac{2n+2}{n!}x^{2n+1}$, 则$S(x)=\left[\sum\limits_{n=1}^{\infty}\dfrac{x^{2n+2}}{n!}\right]'=\left[x^2\cdot\sum\limits_{n=1}^{\infty}\dfrac{(x^2)^n}{n!}\right]'=\left[x^2(\mathrm{e}^{x^2}-1)\right]'=$

$2x(\mathrm{e}^{x^2}-1)+2x^3\mathrm{e}^{x^2},-\infty<x<+\infty$

6.证：解方程$\begin{cases}\dfrac{\partial z}{\partial x}=\left(1+e^{y}\right)\left(-\sin x\right)=0\\ \dfrac{\partial z}{\partial y}=e^{y}\left(\cos x-1-y\right)=0\end{cases}$，

得无穷多个驻点$\left(2k\pi,0\right),\left(2k\pi+\pi,-2\right),k=0,\pm1,\pm2,\dots$

又$\dfrac{\partial^2 z}{\partial x^2}=\left(1+e^{y}\right)\left(-\cos x\right)$,

$\dfrac{\partial^2 z}{\partial x\partial y}=e^{y}\left(-\sin x\right)$,

$\dfrac{\partial^2 z}{\partial y^2}=e^{y}\left(\cos x-1-y\right)-e^{y}$， 8分

于是在点$\left(2k\pi,0\right)$处有$A=-2<,B=0,C=-1$，$B^2-AC=-2<0$

故点$\left(2k\pi,0\right)$是函数的极大值点。

又在点$\left(2k\pi+\pi,-2\right)$处有$A=1+e^{-2},B=0,C=-e^{-2}$，$B^2-AC>0$

故点$\left(2k\pi+\pi,-2\right)$不是函数的极值点。

综上所述，函数有无穷多个极大值点但无极小值点。 10分

7(1)解$:L:x^2+y^2=1$正向$,D:x^2+y^2\le 1,$

取$P(x,y)=-(x^2+y^2)\dfrac{\partial f}{\partial y};Q(x,y)=(x^2+y^2)\dfrac{\partial f}{\partial x};$

则$\oint_L P(x,y)dx+Q(x,y)\mathrm{d}y=\iint\limits_D(\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y})\mathrm{d}x\mathrm{d}y$

$=2\iint\limits_D(x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y})\mathrm{d}x\mathrm{d}y+\iint\limits_D(x^2+y^2)(\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2})\mathrm{d}x\mathrm{d}y$.....................(1)

又$\oint_L P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y=\oint_L -(x^2+y^2)\dfrac{\partial f}{\partial y}\mathrm{d}x+(x^2+y^2)\dfrac{\partial f}{\partial x}\mathrm{d}y$

$=\oint_L -\dfrac{\partial f}{\partial y}\mathrm{d}x+\dfrac{\partial f}{\partial x}\mathrm{d}y=\iint\limits_D(\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2})\mathrm{d}x\mathrm{d}y$.......................................(2)

(1)\,(2)两式相减得

$$\iint\limits_D(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y})\mathrm{d}x\mathrm{d}y=\frac{1}{2}\left[\iint\limits_D(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2})\mathrm{d}x\mathrm{d}y-\iint\limits_D(x^2+y^2)(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2})\mathrm{d}x\mathrm{d}y\right]$$

$$=\frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{e}^{-r^2}r\mathrm{d}r-\frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^3\mathrm{e}^{-r^2}\mathrm{d}r=\frac{\pi}{2}(1-\frac{1}{\mathrm{e}})-(1-\frac{\pi}{\mathrm{e}})=\frac{\pi}{2}+\frac{\pi}{\mathrm{e}}-1.$$

(2)解：级数$\sum_{n=1}^{\infty}\arctan\frac{1}{2n^2}$的一般项$u_n=\arctan\frac{1}{2n^2}$，当$n\to\infty$时，

$\arctan\frac{1}{2n^2}\sim\frac{1}{2n^2}$,用比较审敛法的极限形式，因为$\lim_{n\to\infty}\frac{\arctan\frac{1}{2n^2}}{\frac{1}{2n^2}}=1$,

又级数$\sum_{n=1}^{\infty}\frac{1}{2n^2}$收敛，所以$\sum_{n=1}^{\infty}\arctan\frac{1}{2n^2}$也收敛。

下面求$\sum_{n=1}^{\infty}\arctan\frac{1}{2n^2}$的和$S$，$S_1=\arctan\frac{1}{2}$,

$S_2=u_1+u_2=\arctan\frac{1}{2}+\arctan\frac{1}{8}=\arctan\frac{\frac{1}{2}+\frac{1}{8}}{1-\frac{1}{2}\bullet\frac{1}{8}}=\arctan\frac{2}{3}$,

$S_3=S_2+u_3=\arctan\frac{2}{3}+\arctan\frac{1}{18}=\arctan\frac{\frac{2}{3}+\frac{1}{18}}{1-\frac{2}{3}\bullet\frac{1}{18}}=\arctan\frac{3}{4}$,

…，由数学归纳法得到，$S_n=\arctan\frac{n}{n+1}$,

从而S=$\lim_{n\to\infty}\arctan\frac{n}{n+1}=\arctan 1=\frac{\pi}{4}$，所以$\sum_{n=1}^{\infty}\arctan\frac{1}{2n^2}=\frac{\pi}{4}$。