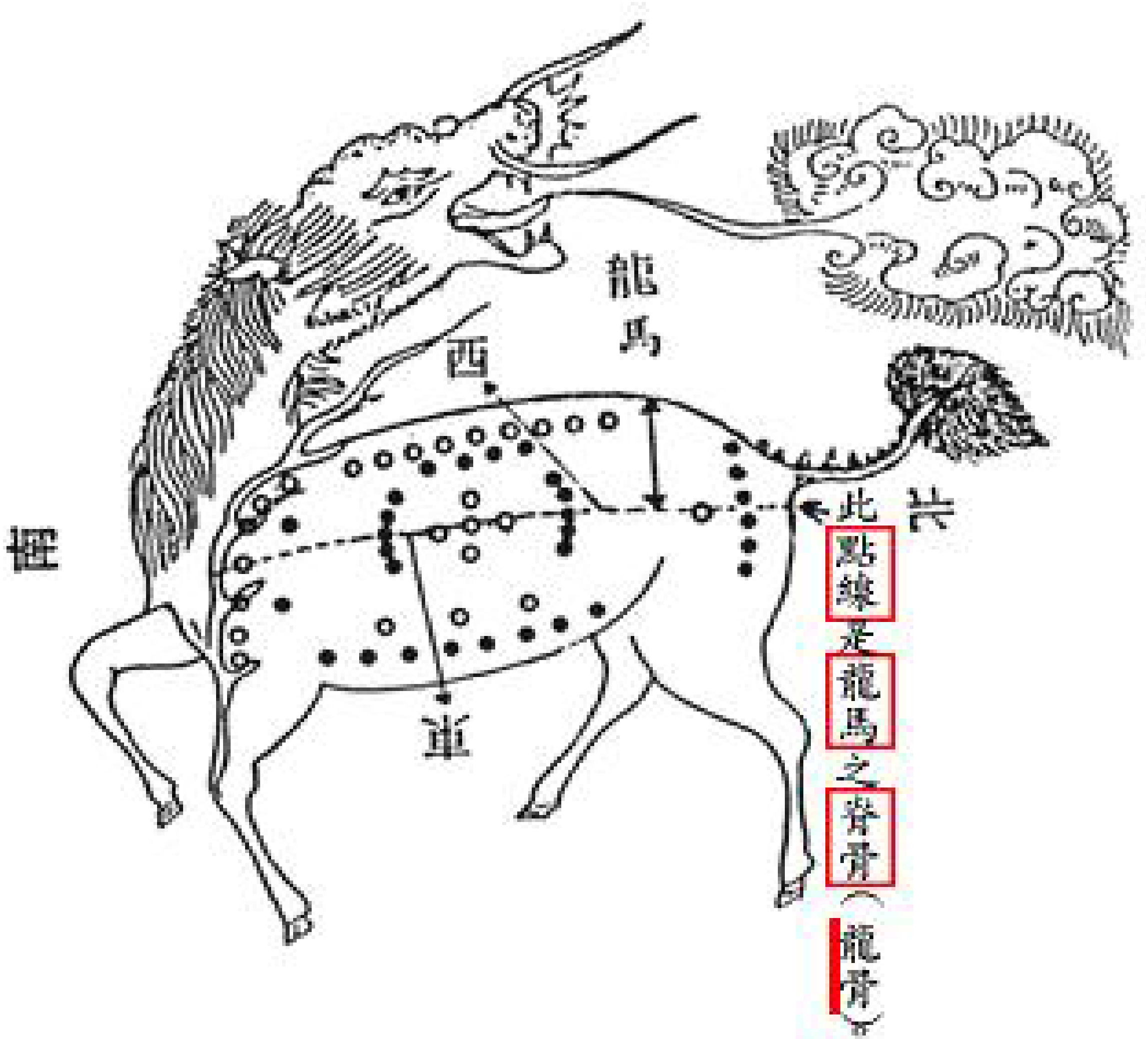
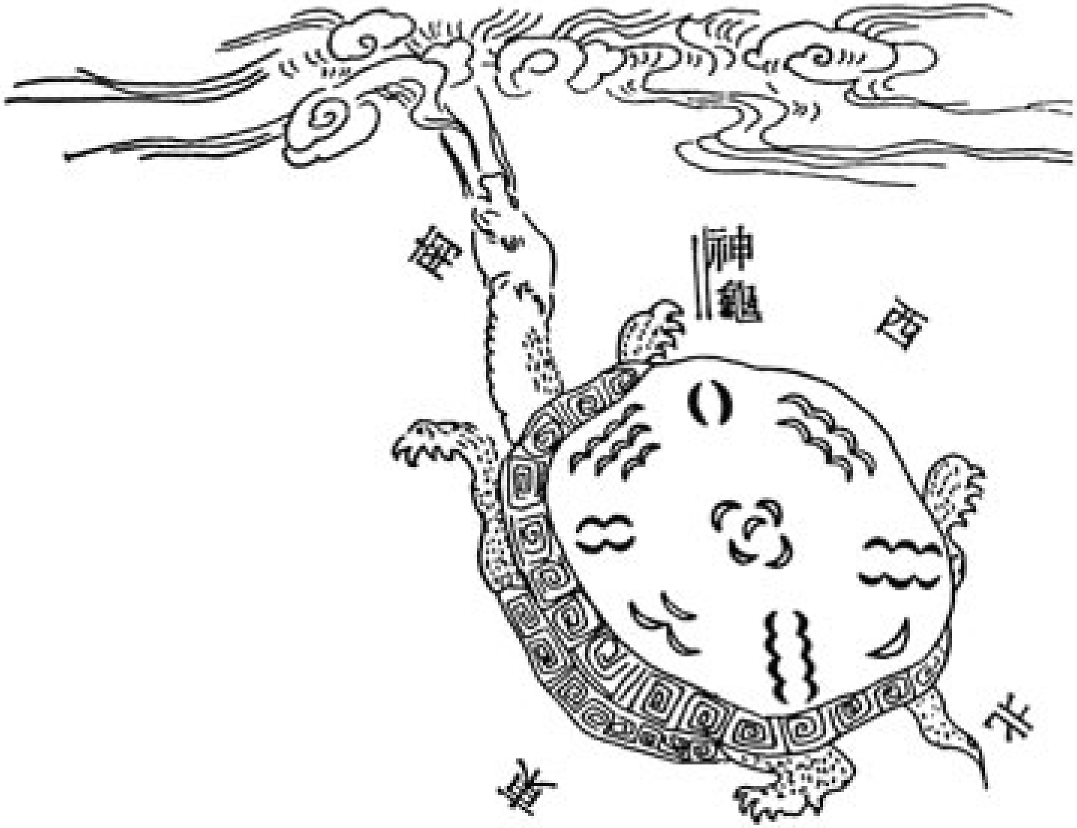
Chapter 6

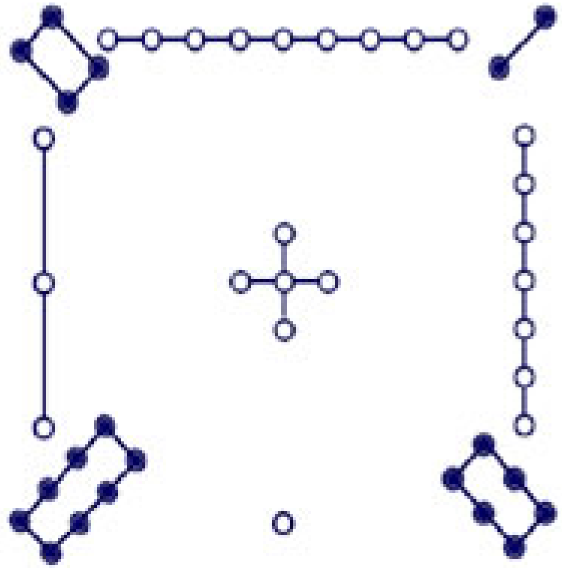
Counting

2015-11-3

magic square 幻方







洛书

Magic Square

4	9	2	15
3	5	7	15
8	1	6	15
15	15	15	15

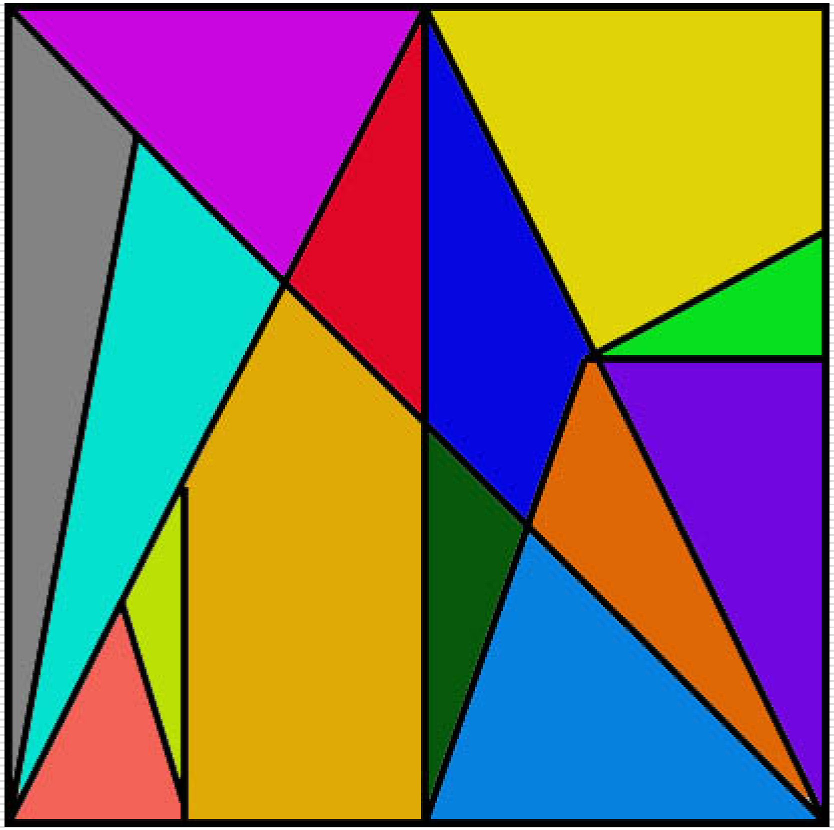
四阶幻方

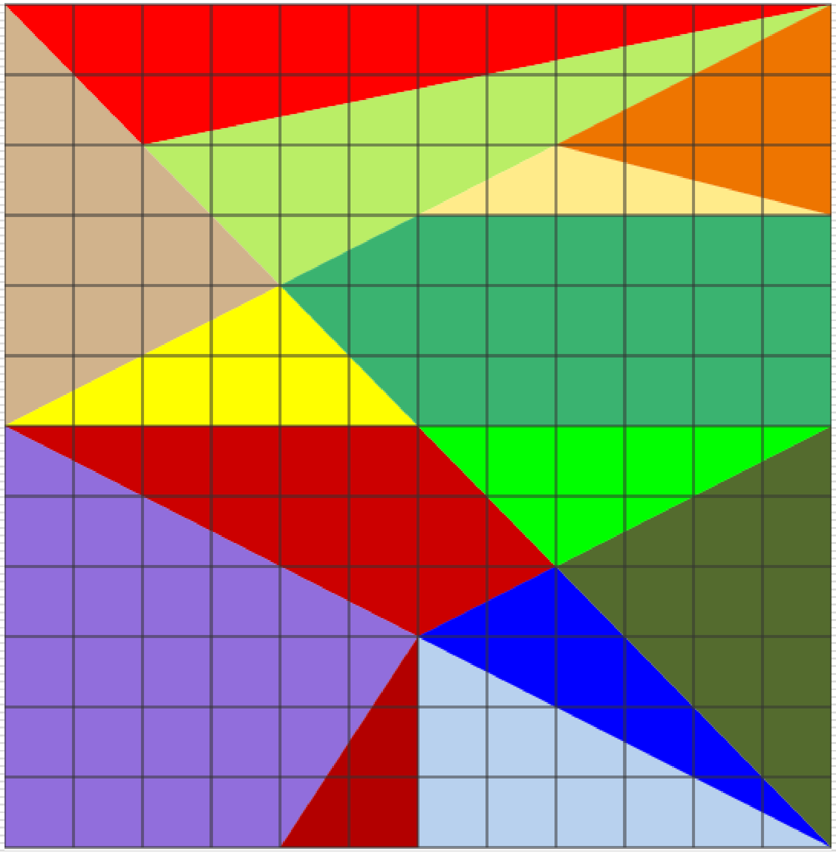
1	**15**	**14**	**4**
12	**6**	**7**	**9**
8	**10**	**11**	**5**
13	**3**	**2**	**16**

阿基米德手稿

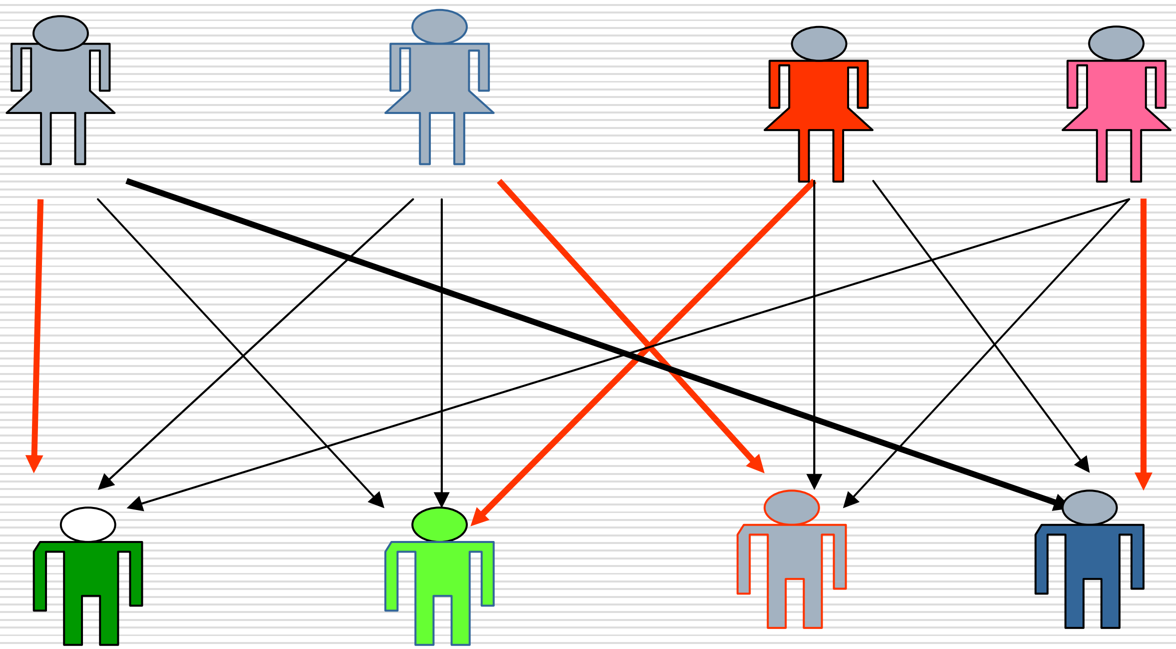
上图为一份用希腊文写在羊皮纸上的阿基米德手稿副本，最近科学家借助现代科技手段初步破译了古希腊数学家阿基米德的这篇论文，结论是这篇被称作 Stomachion 的论文解决的是组合数学问题。

Stomachion

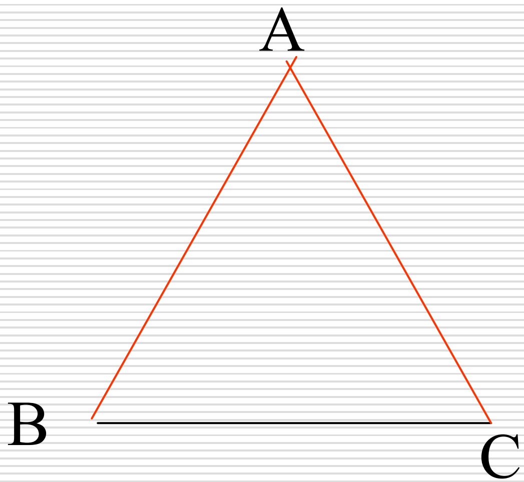




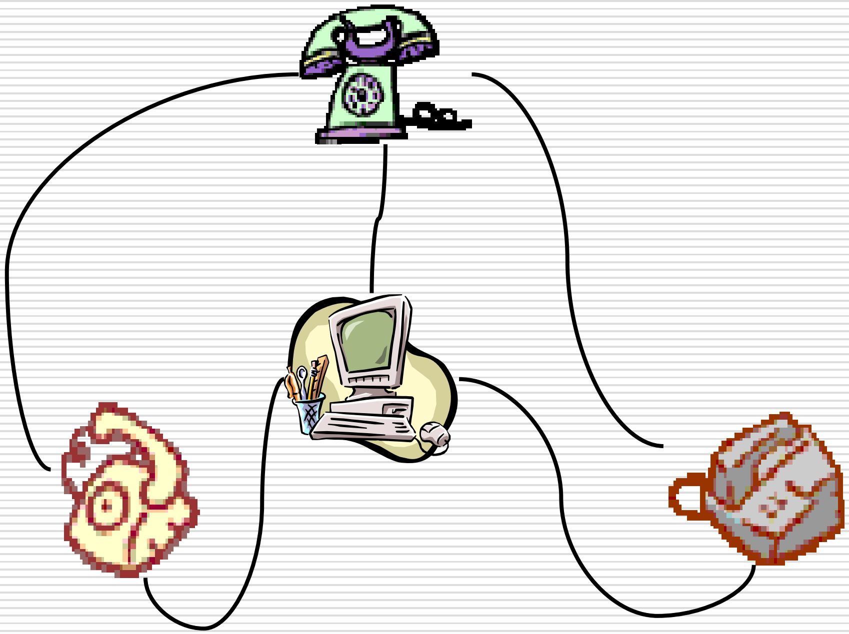
稳定婚姻问题

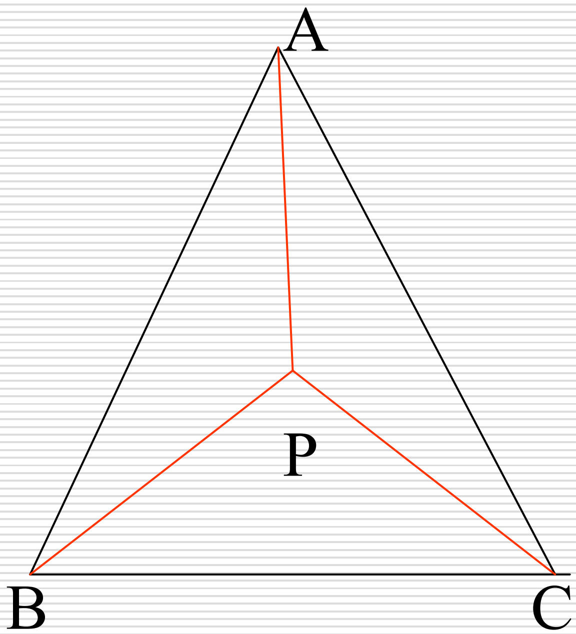


最短网络问题



最短网络问题





Chapter Summary

- The Basics of Counting
- The Pigeonhole Principle
- Permutations and Combinations
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations

§ 6.1 The basics of counting

Section Summary

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Tree Diagrams

§6.1 The basics of counting (1)

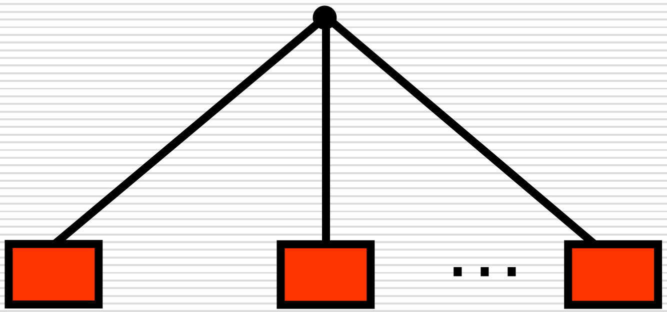
6.1.1 Basic counting principles

（1） The product rule

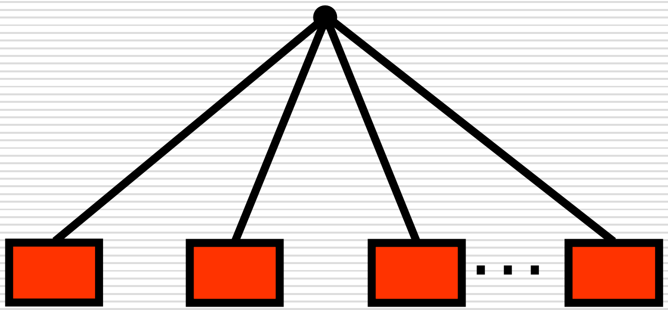
Definition:

Suppose that a procedure can be broken down into a sequence of two tasks.If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done,then there are $n_1 \times n_2$ ways to do the procedure.

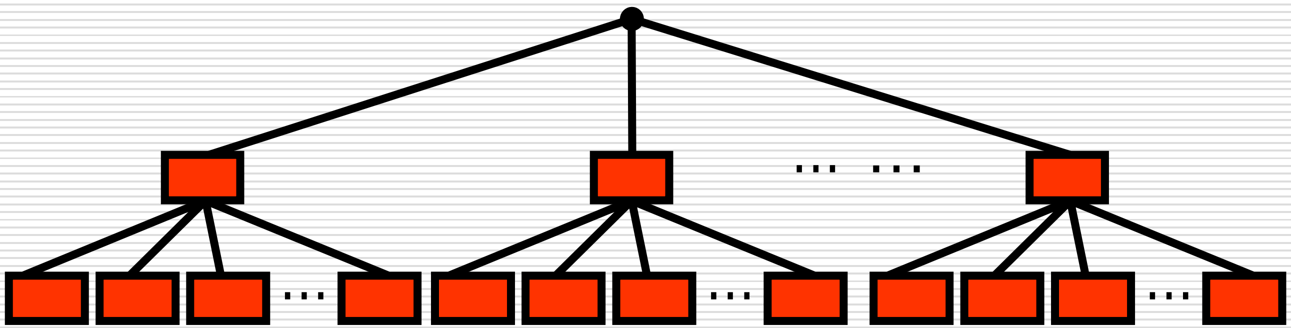
§ 6.1 The basics of counting (2)



完成任务T_1的可能方法



完成任务T_2的可能方法



先执行任务T_1再执行任务T_2的可能方法

§ 6.1 The basics of counting (3)

6.1.1 Basic counting principles

（1）The product rule

An extended version of the product rule:

Suppose that a procedure is carried out by performing the tasks $T_1,T_2,\ldots,T_m$ in sequence. If task T_i can be done in n_i ways after tasks $T_1,T_2,\ldots,$and T_{i-1} have been done,then there are $n_1\times n_2\times\ldots\times n_m$ ways to carry out the procedure.

§6.1 The basics of counting (4)

6.1.1 Basic counting principles

（2） The sum rule

Definition:

If a first task can be done in n_1 ways and a second task in n_2 ways, and if these tasks cannot be done at the same time, then there are n_1+n_2 ways to do one of these tasks.

§ 6.1 The basics of counting (5)

6.1.1 Basic counting principles

（2）The sum rule

An extended version of the sum rule:

Suppose that the tasks $T_1, T_2, \ldots, T_m$ can be done in $n_1, n_2, \ldots, n_m$ ways,respectively,and no two of this tasks can be done at the same time.Then the number of ways to do one of these tasks is $n_1+n_2+\ldots+n_m$.

§ 6.1 The basics of counting (6)

6.1.2 More complex counting problems

6.1.3 The inclusion-exclusion principle

6.1.4 The Division Rule

6.1.5 Tree diagrams

§6.2 The Pigeonhole principle (1)

6.2.1 Introduction

The pigeonhole principle:

If $k+1$ or more objects are placed into k boxes ,then there is at least one box containing two or more of the objects.

§ 6.2 The Pigeonhole principle (2)

6.2.2 The generalized Pigeonhole principle

The generalized Pigeonhole principle:

If N objects are placed into K boxes, then there is at least one box containing at least $\lceil N/K \rceil$ objects.

§6.2 The Pigeonhole principle (3)

6.2.2 The generalized Pigeonhole principle

例：最少连接线数目的问题

条件：15个工作站和10台服务器。每个工作站可以用一条线直接连接到某台服务器上。同一时刻每台服务器只能接受一个工作站的访问。

目标：任何时刻、任何一个工作站可以通过直接连线访问一台服务器。

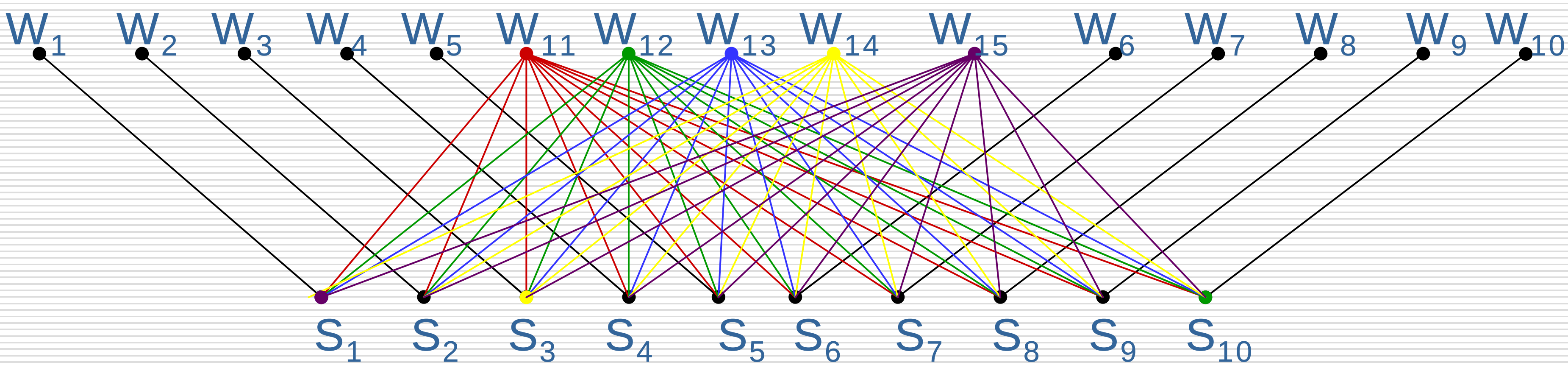
即：同一时刻一台服务器只能为一个工作站提供服务

问题：达到这个目标需要的最少接线数目是多少？

§ 6.2 The Pigeonhole principle (4)

假设将工作站标记为：$W_1, W_2, \ldots, W_{15}$，服务器标记为$S_1, S_2, \ldots, S_{10}$。

对于$k=1,2,\ldots,10$，连接W_k到S_k，剩下 5 个工作站的每一个都连接到10 台服务器。总共60 条直接连线.



§ 6.2 The Pigeonhole principle (5)

6.2.3 Some elegant applications of the pigeonhole principle

Theorem:

Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

12, 8, 7, 9, 10, 1, 4, 11, 6, 5

5, 3, 2, 4, 1

§ 6.2 The Pigeonhole principle (6)

6.2.3 Some elegant applications of the pigeonhole principle

定义：假设p, q为正整数，令R(p, q)是保证有p个人是彼此相识或者有 q 个人彼此不相识所需要的最少的人数，则称R(p, q)为Ramsey数。

R(3,3) = 6， R(3,4) = 9， R(3,5) = 14
R(3,6) = 18，R(3,7) = 23，R(3,8) = 28
R(3,9) = 36，R(4,4) = 18。

§6.2 The Pigeonhole principle (7)

6.2.3 Some elegant applications of the pigeonhole principle

定理： 设 p, q 正整数，p, q ≥ 2，则存在最小正整数R(p, q)，使得当n ≥ R(p,q) 时，用红蓝两色涂色 K_n 的边，则或存在一个蓝色的K_p，或存在一个红色的K_q。

§6.3 Permutations and Combinations (1)

6.3.1 Permutations

（1） r-permutation

Definition:

A permutation of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an r-permuation.

§ 6.3 Permutations and Combinations (2)

6.3.1 Permutations

（1） r-permutation

The number of r-permutations of a set with n elements is denoted by P(n,r).

$$P(n,r) = \begin{cases} 1 & n \geq r = 0 \\ 0 & n < r \end{cases}$$

§ 6.3 Permutations and Combinations (3)

6.3.1 Permutations

（1） r-permutation

Theorem：

The number of r-permutations of a set with n distinct elements is

P(n,r) = n(n-1)(n-2)….(n-r+1).

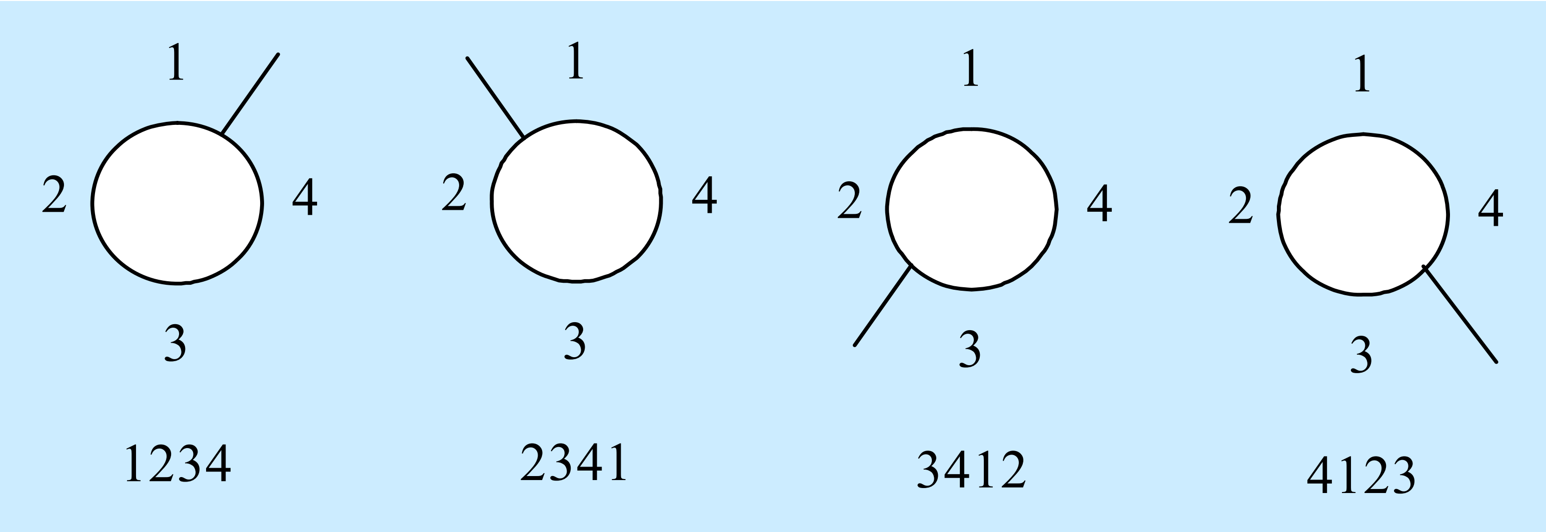
$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

$$P(n,n) = n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

§ 6.3 Permutations and Combinations (4)

6.3.1 Permutations

（2）Circle-permutation

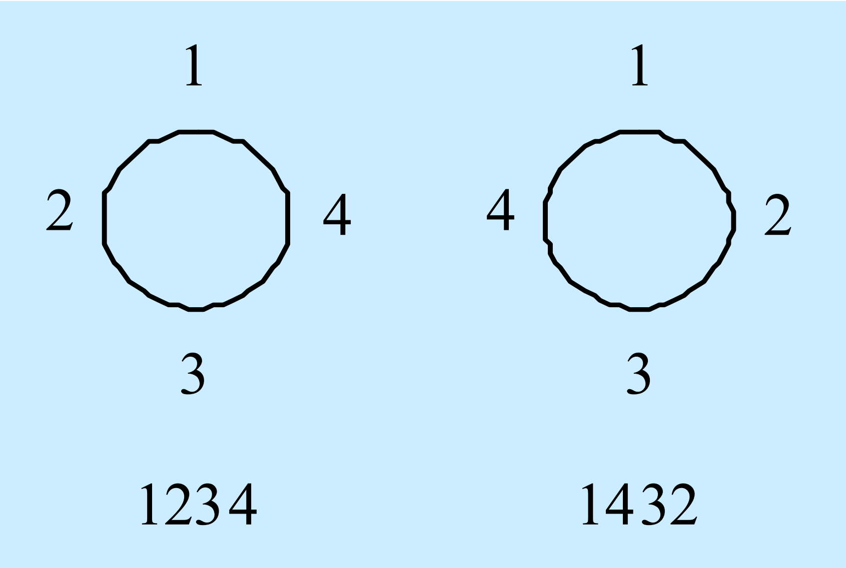


$$P(n,r)/r = n!/(r(n-r)!)$$

§6.3 Permutations and Combinations (5)

6.3.1 Permutations

（2）Circle-permutation



$$P(n,r)/(2r)$$

6.3.2 Combinations

r-combination

An r-combination of elements of a set is an unordered selection of r elements from the set.Thus, an r-combination is simply a subset of the set with r elements.

The number of r-combination of a set with n distinct elements is denoted by C(n,r).

§ 6.3 Permutations and Combinations (7)

6.3.2 Combinations

r-combination

Note that C(n,r) is also denoted by $\binom{n}{r}$ and is called a binomial coefficient.

$$C(n,r) = \begin{cases} 1 & n \geq r = 0 \\ 0 & n < r \end{cases}$$

6.3.2 Combinations

r-combination

Theorem:

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

§ 6.3 Permutations and Combinations (9)

6.3.2 Combinations

r-combination

Corollary:

Let n and r be nonnegative integers with $r \le n$ .Then $C(n,r) = C(n,n-r)$.

§ 6.4 Binomial coefficients and Identities (1)

6.4.1 The binomial theorem

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

$$(x+y)^0 = 1$$

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

§ 6.4 Binomial coefficients and Identities（2）

n \ r	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

§ 6.4 Binomial coefficients and Identities (3)

6.4.1 The binomial theorem

The binomial theorem: Let x and y be variables,and let n be a nonnegative integer.Then

$$(x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^{j}$$

$$=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n$$

§ 6.4 Binomial coefficients and Identities (4)

6.4.1 The binomial theorem

Example 1:

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^{j}$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^{4-1} y + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

§ 6.4 Binomial coefficients and Identities (5)

6.4.1 The binomial theorem

Example 2:

$$(2x+3y)^4 = \sum_{j=0}^{4} \binom{4}{j} (2x)^{4-j} (3y)^j$$

$$= \binom{4}{0} 16x^4 + \binom{4}{1} 24x^3 y + \binom{4}{2} 36x^2 y^2 + \binom{4}{3} 54xy^3 + \binom{4}{4} 81y^4$$

$$= 16x^4 + 96x^3 y + 216x^2 y^2 + 216xy^3 + 81y^4$$

§ 6.4 Binomial coefficients and Identities (6)

6.4.1 The binomial theorem

Example 3: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

$$\binom{25}{13} = \frac{25!}{13!12!} = 5{,}200{,}300$$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

§ 6.4 Binomial coefficients and Identities (7)

6.4.1 The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^{k} y^{n-k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{k} y^{n-k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^{k}$$

§ 6.4 Binomial coefficients and Identities (8)

6.4.1 The binomial theorem

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k}$$

$$= \sum_{k=0}^{n} \binom{n}{n-k} x^{k}$$

§ 6.4 Binomial coefficients and Identities (9)

6.4.1 The binomial theorem

Corollary 1: Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Corollary 2: Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Corollary 3: Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

§ 6.4 Binomial coefficients and Identities(10)

6.4.2 Pascal's identity and triangle

Pascal's identity: Let n and k be positive integers with n ≥ k .Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

6.4.3 Some other identities of the binomial coefficients

Vandermonde's identity:

Let m,n and r be nonnegative integers with r not exceeding either m or n,Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

6.4.3 Some other identities of the binomial coefficients

Corollary 4: If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

6.4.3 Some other identities of the binomial coefficients

Theorem: Let n and r be nonnegative integers with $r \le n$ .Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

§ 6.5 Generalized permutation and combinations (1)

6.5.1 Permutations with Repetition

Theorem:

The number of r-permutations of a set of n objects with repetition allowed is n^r.

§6.5 Generalized permutation and combinations (2)

6.5.2 Combinations with Repetition

Theorem:

There are $C(n+r-1,r)$ r-combinations from a set with n elements when repetition of elements is allowed.

§ 6.5 Generalized permutation and combinations (3)

6.5.3 Permutations with indistinguishable objects

Theorem: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

§ 6.5 Generalized permutation and combinations (4)

6.5.4 Distributing objects into boxes

（1）Distinguishable Objects and Distinguishable Boxes

Theorem:

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i (i=1,2,..,k),equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

§6.5 Generalized permutation and combinations (5)

6.5.4 Distributing objects into boxes

（2）Indistinguishable Objects and Distinguishable Boxes

（3）Distinguishable Objects and Indistinguishable Boxes

（4）Indistinguishable Objects and Indistinguishable Boxes

§6.6 Generating permutations and combinations (1)

6.6.1 Generating permutations

Lexicographic(Dictionary) ordering

Example:

字符集{1,2,3},较小的数字较先,这样按字典序生成的全排列是:

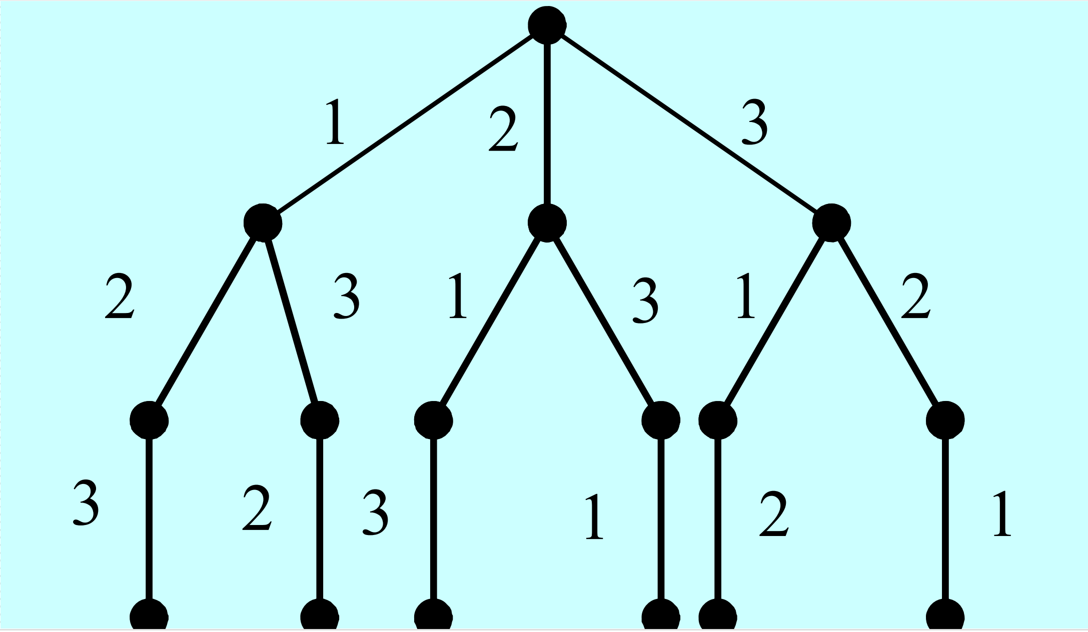
123,132,213,231,312,321。

※ 一个全排列可看做一个字符串，字符串可有**前缀、后缀**。

§ 6.6 Generating permutations and combinations (2)

6.6.1 Generating permutations

Lexicographic(Dictionary) ordering



§6.6 Generating permutations and combinations (3)

6.6.1 Generating permutations

Lexicographic(Dictionary) ordering

生成给定全排列的下一个排列，所谓一个排列的下一个排列就是这一个排列与下一个排列之间没有其他的排列。这就要求这一个与下一个有尽可能长的共同前缀，也即变化限制在尽可能短的后缀上。

§6.6 Generating permutations and combinations (4)

6.6.1 Generating permutations

例如：给定排列 $a_1\ a_2\ \ldots\ a_n$，若 $a_{n-1} < a_n$

则它的下一个排列为： $a_1\ a_2\ \ldots\ a_n\ a_{n-1}$

例如：给定序列 1 2 3 4 5，它的下一个排列

1 2 3 5 4

若 $a_{n-1} > a_n$ 则不能进行交换产生更大的排列

若 $a_{n-2} < a_{n-1}$ 是否可以交换产生下一个排列？

§6.6 Generating permutations and combinations (5)

6.6.1 Generating permutations

Arithmetic:

(1) $i = \max\{k \mid a_{k-1} < a_k\}$

(2) $j = \max\{k \mid a_{i-1} < a_k\}$

(3) $a_{i-1} \leftrightarrow a_j$

(4) $a_1, a_2, \ldots, a_j, a_i, a_{i+1}, \ldots a_{i-1} \ldots, a_n$

$a_1, a_2, \ldots, a_{i-1}, a_n, a_{n-1}, \ldots, a_i$

§ 6.6 Generating permutations and combinations (6)

6.6.1 Generating permutations

例如：给定排列 8 3 9 6 4 7 5 2 1

⇒ 8 3 9 6 4 7 5 2 1 ⇒ 8 3 9 6 4 7 5 2 1

⇒ 8 3 9 6 5 7 4 2 1 ⇒ 8 3 9 6 5 1 2 4 7

⇒ 8 3 9 6 5 1 2 7 4

⇒ 8 3 9 6 5 1 4 2 7

⇒ 8 3 9 6 5 1 4 7 2

§6.6 Generating permutations and combinations (7)

6.6.2 Generating combinations

Example:字符集{1,2,3,4,5,6}取3元素的组合。

共有20个

123，124，125，126
134，135，136
145，146
156
234，235，236
245，246
256
345，346
356
456

§ 6.6 Generating permutations and combinations (8)

6.6.2 Generating combinations

Arithmetic:

(1) If $\exists\, i = \max\{j \mid c_j < n-r+j\}$

Then goto (2)

Else stop ;

(2) $c_i \leftarrow c_i + 1$;

(3) $c_j \leftarrow c_{j-1} + 1, j=i+1, i+2, \ldots, r$;

(4) Output $c_1 c_2 \ldots c_r$; goto (1);

Chapter 8

Advanced counting techniques

Chapter Summary

- Applications of Recurrence Relations
- Solving Linear Recurrence Relations
  - Homogeneous Recurrence Relations
  - Nonhomogeneous Recurrence Relations
- Divide-and-Conquer Algorithms and Recurrence Relations
- Generating Functions
- Inclusion-Exclusion
- Applications of Inclusion-Exclusion

§ 8.1 Applications of Recurrence Relations (1)

8.1.1 The concept of recurrence relations

Defination:

A sequence is a function from a subset of the set of integers (usually either the set {0,1,2,...} or the set {1,2,3,...}) to a set S.We use the notation a_n to denote the image of the integer n.We call a_n a term of the sequence.

We use the notation $\{a_n\}$ to describe the sequence.

§ 8.1 Applications of Recurrence Relations (2)

8.1.1 The concept of recurrence relations

Defination:

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence,namely,a_0,a_1,.., a_{n-1},for all integers n with $n \geq n_0$,where n_0 is a nonnegative integer.A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

§ 8.1 Applications of Recurrence Relations (3)

8.1.2 Modeling with recurrence relations

Example 1: Bacterial Reproduction

$$a_n = 2a_{n-1},\ \ a_0 = 5;\ \ a_n = 2^n \times 5$$

Example 2: Compound Interest

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

$$P_n = (1.05)^n 1000$$

§ 8.1 Applications of Recurrence Relations (4)

8.1.2 Modeling with recurrence relations

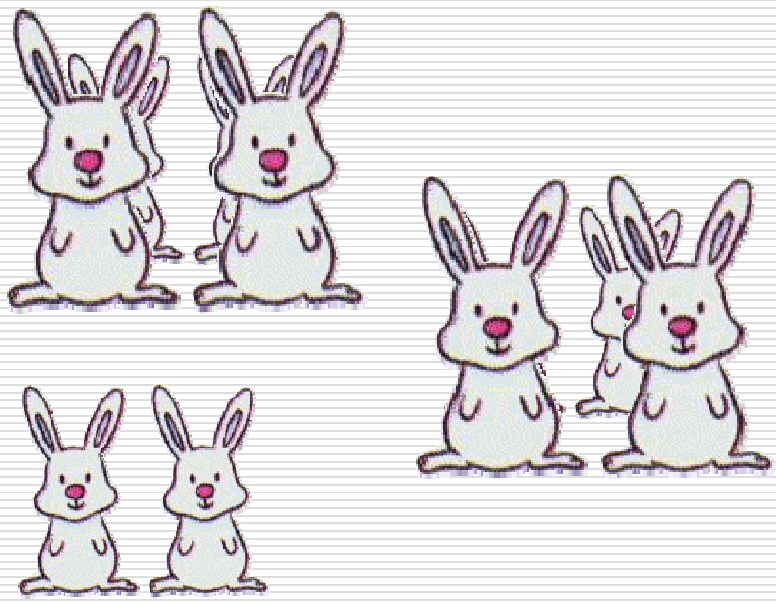
Example 3：Fibonacci Numbers

第1个月 1 对

第2个月 1 对

第3个月 2 对

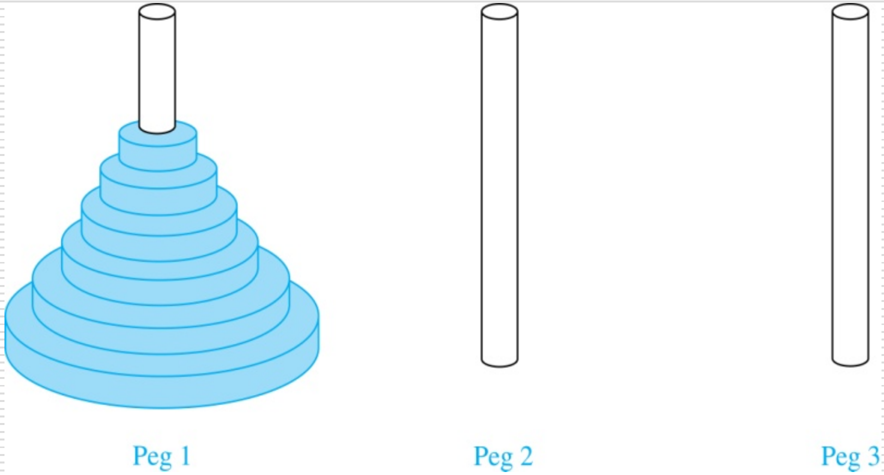
第4个月 3 对

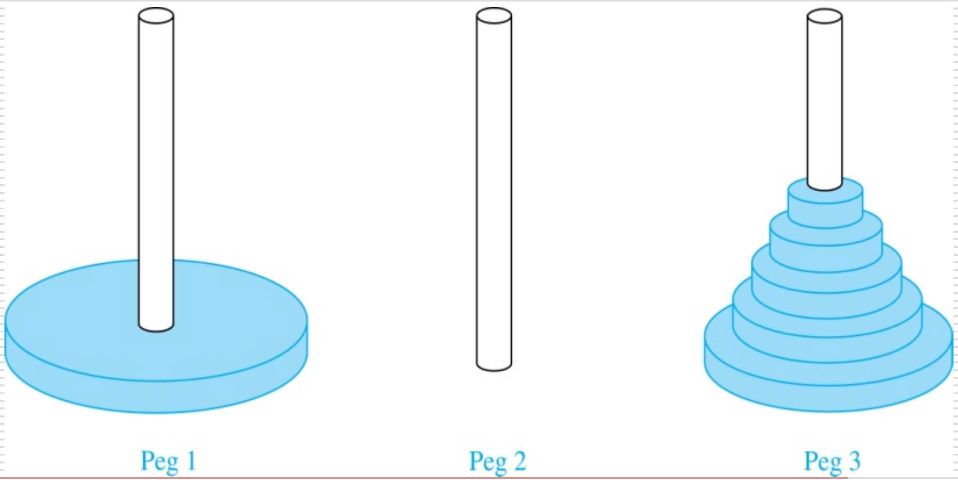


§8.1 Applications of Recurrence Relations (5)

8.1.2 Modeling with recurrence relations

Example 4: The Tower Hanoi





§8.1 Applications of Recurrence Relations (6)

8.1.2 Modeling with recurrence relations

Example 5： Bit Strings

Example 6： Codeword Enumeration

Example 7： Lancaster

§ 8.2 Solving Linear Recurrence Relations (1)

8.2.1 Linear homogeneous recurrence relation of degree k

Definition:

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers , and $c_k \neq 0$.

§ 8.2 Solving Linear Recurrence Relations (2)

8.2.1 Linear homogeneous recurrence relation of degree k

$F_n = F_{n-1} + F_{n-2}$ linear homogeneous recurrence relation of degree two

$a_n = a_{n-1} + a_{n-2}^2$ not linear

$H_n = 2H_{n-1} + 1$ not homogeneous

$B_n = nB_{n-1}$ coefficients are not constants

§ 8.2 Solving Linear Recurrence Relations (3)

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

Linear homogeneous recurrence relation with constant coefficients :

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0$$

Characteristic equation:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

（1）distinct root

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2-c_1r-c_2=0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}$ if and only if $a_n=b_1r_1^n+b_2r_2^n$ for $n=0,1,2,\ldots$, where b_1 and b_2 are constants.

§ 8.2 Solving Linear Recurrence Relations (5)

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

（1）distinct root

$$\begin{cases} a_0 = C_0 = b_1 + b_2 \\ a_1 = C_1 = b_1 r_1 + b_2 r_2 \end{cases}$$

$$b_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

$$b_2 = C_0 - b_1$$

$$= C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2}$$

§8.2 Solving Linear Recurrence Relations (6)

Example 1:求解下列递推关系的解。

$$\begin{cases} a_n = a_{n-1} + 2a_{n-2} \\ a_0 = 2, a_1 = 7 \end{cases}$$

Example 2:求解Fibonacci递推关系的解。

$$\begin{cases} F_n = F_{n-1} + F_{n-2} & n \geq 3 \\ F_1 = F_2 = 1 \end{cases}$$

Example 3:

只有三个字母a,b,c组成的长度为n的一些单词将在通信信道上传输。必须满足下列条件：传输中不得有两个a连续出现在任一单词中。确定通信信道允许传输的单词个数。

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

（1） distinct root

Theorem 2: Let c1,c2,…,ck be real numbers. Suppose that the characteristic equation $r^k-c_1r^{k-1}-\ldots-c_k=0$ has k distinct roots $r_1,r_2,\ldots,r_k$.Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}$ if and only if $a_n=b_1r_1^n+b_2r_2^n+\ldots+b_kr_k^n$, for n=0,1,2,…, where $b_1,b_2,\ldots,b_k$ are constants.

§ 8.2 Solving Linear Recurrence Relations (9)

Example: 求下列递推关系的解。

$$\begin{cases} a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} & (n \geq 3) \\ a_0 = 1, a_1 = 2, a_2 = 0 \end{cases}$$

8.2.2 Solving linear homogeneous recurrence relation with constant coefficients

（2） Multiple root

Theorem 3: Let c_1 and c_2 be real numbers wth $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = b_1 r_0^n + b_2 n r_0^n$ for $n = 0, 1, 2, \ldots$, where b_1 and b_2 are constants.

§8.2 Solving Linear Recurrence Relations (11)

Example：求解下列递推关系的解。

$$\begin{cases} a_n = 2a_{n-1} - a_{n-2} \quad (n \geq 3) \\ a_1 = 2, a_2 = 3 \end{cases}$$

§ 8.2 Solving Linear Recurrence Relations (12)

Theorem 4: Let c1,c2,…,ck be real numbers. Suppose that the characteristic equation $r^k-c_1r^{k-1}-\ldots-c_k=0$ has t distinct roots $r_1,r_2,\ldots,r_t$ with multiplicities $m_1,m_2,\ldots,m_t$, respectively ,so that $m_i \geq 1$ for $i=1,2,\ldots,t$ and $m_1+m_2+\ldots+m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}$ if and only if

$$a_n = (b_{1,0}+b_{1,1}n+\ldots+b_{1,m_1-1}n^{m_1-1})r_1^n$$
$$+(b_{2,0}+b_{2,1}n+\ldots+b_{2,m_2-1}n^{m_2-1})r_2^n$$
$$+\ldots+(b_{t,0}+b_{t,1}n+\ldots+b_{t,m_t-1}n^{m_t-1})r_t^n$$

for $n=0,1,2,3,\ldots$, where $b_{i,j}$ are constants for $1\leq i \leq t$ and $0 \leq j \leq m_i-1$.

§8.2 Solving Linear Recurrence Relations (13)

Example：求解下述递推关系的解。

$$\begin{cases} a_n = -a_{n-1} + 3a_{n-2} + 5a_{n-3} + 2a_{n-4} \quad (n \geq 4) \\ a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 2 \end{cases}$$

Solving linear homogeneous recurrence relation with constant coefficients

（1）The characteristic equation

（2）The characteristic roots

（3）Distinct root？ Multiple root？

（4）The initial conditions

§ 8.2 Solving Linear Recurrence Relations (14)

Example：求解下述递推关系的解。

$$\begin{cases} a_n = 8a_{n-2} - 16a_{n-4} & (n \geq 4) \\ a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 2 \end{cases}$$

特征方程：$r^4 - 8r^2 + 16 = 0$

特征方程的根：$r_1 = 2$，$r_2 = -2$ 都是二重根

递推关系的解：$a_n = (b_1 + b_2 n) r_1^n + (b_3 + b_4 n) r_2^n$

8.2.3 Linear nonhomogeneous recurrence relations with constant coefficients

A **linear nonhomogeneous recurrence relation with constant coefficients**
is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

where $c_1, c_2, \ldots c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n.
The recurrence relaton

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$, is called the **associated homogeneous recurrence relation .**

§ 8.2 Solving Linear Recurrence Relations (14)

Example:

$$a_n = a_{n-1} + 2^n$$

$$a_n = a_{n-1}$$

$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$$

$$a_n = a_{n-1} + a_{n-2}$$

$$a_n = 3a_{n-1} + n3^n$$

$$a_n = 3a_{n-1}$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

8.2.3 Linear nonhomogeneous recurrence relations with constant coefficients

Theorem 1:

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence ralation with constant coefficients

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}+F(n),$$

then every solutionis of the form $\{a_n^{(p)}+a_n^{(h)}\}$,where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence ralation

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}.$$

§ 8.2 Solving Linear Recurrence Relations (16)

$$a_n^{(h)} = c_1 a_{n-1}^{(h)} + c_2 a_{n-2}^{(h)} + \cdots + c_k a_{n-k}^{(h)}$$

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n)$$

$$a_n^{(h)} + a_n^{(p)} = c_1(a_{n-1}^{(h)} + a_{n-1}^{(p)}) + c_2(a_{n-2}^{(h)} + a_{n-2}^{(p)}) + \cdots$$
$$+ c_k(a_{n-k}^{(h)} + a_{n-k}^{(p)}) + F(n)$$

$$a_n = a_n^{(p)} + a_n^{(h)} = a_n^{(p)} + \sum_{i=1}^{k} b_i r_i^n$$

§8.2 Solving Linear Recurrence Relations (17)

初始条件：$a_0 = C_0, a_1 = C_1, \ldots\ldots, a_{k-1} = C_{k-1}$

$$b_1 + b_2 + \cdots + b_k = C_0 *$$

$$r_1 b_1 + r_2 b_2 + \cdots + r_k b_k = C_1 *$$

$$\cdots\cdots$$

$$r_1^{k-1} b_1 + r_2^{k-1} b_2 + \cdots + r_k^{k-1} b_k = C_{k-1} *$$

其中：$C_i^* = C_i - a_i^{(p)}$

§8.2 Solving Linear Recurrence Relations (18)

（1）当F(n)是 n 的 k 次多项式时，可设递推关系的特解形式为：

$$a^{(p)} = A_0 n^k + A_1 n^{k-1} + \cdots + A_k$$

式中$A_0, A_1, \ldots, A_k$为待定常数。

Example 1：求解下列递推关系

$$\begin{cases} a_n = -2a_{n-1} + n + 3 \\ a_0 = 3 \end{cases}$$

§8.2 Solving Linear Recurrence Relations (19)

当相伴递推关系的特征根为 1 的 m （m ≥ 1）重根时，特解的形式为:

$$a^{(p)} = (A_0 n^k + A_1 n^{k-1} + \cdots + A_k)\ n^m$$

式中 $A_0, A_1, \ldots, A_k$ 为待定常数。

Example 2: 求解下列递推关系

$$\begin{cases} a_n = a_{n-1} + 2(n-1) & (n \geq 2) \\ a_1 = 2 \end{cases}$$

§8.2 Solving Linear Recurrence Relations (20)

（2）当F(n)是β^n的形式时，又可分为如下两种情形：

I、如果β不是相伴递推关系式的特征根时，可设特解的形式为： $a_n^{(p)} = A\beta^n$

II、如果β是相伴递推关系式的k重特征根时（k≥1），可设特解的形式为：$a_n^{(p)} = n^k A\beta^n$

（3）当F(n)是 $P_m(n)\beta^n$ 的形式时

$$a_n^{(p)} = n^k (A_0 n^m + A_1 n^{m-1} + \cdots + A_m)\beta^n$$

§ 8.2 Solving Linear Recurrence Relations (21)

Theorem 2:

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \dots c_k$ are real numbers and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0)\beta^n,$$

where $b_0, b_1, \dots, b_t$ and β are real numbers.

When β is not a root of the characteristic equation of the associated linear homogeneous relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0)\,\beta^n.$$

When β is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0)\,\beta^n.$$

§ 8.2 Solving Linear Recurrence Relations (22)

Example

假设线性非齐次递推关系为:

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

当 $F(n) = 3^n$　$F(n) = n3^n$　$F(n) = n^2 2^n$

$F(n) = (n^2 + 1)3^n$

时，上述线性非齐次递推关系的特解是什么？

§ 8.3 Divide-and-Conquer Algorithms and Recurrence relations (1)

Suppose that a recursive algorithm divides a problem of size n into t subproblems,where each subproblem is of size n/b.Also suppose that a total of g(n) extra operations are required in the conquer step of the algorithm to combine the solutions of the subproblems into a solution of the original problem.Then ,if f(n) represents the number of operations required to solve the problem of size n,it follows that f satisfies the recurrence relation

$$f(n) = t\, f(n/b) + g(n).$$

This is called a divide-and-conquer recurrence relation.

§ 8.3 Divide-and-Conquer Algorithms and Recurrence relations (2)

Theorem 1:

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a\, f(n/b) + c$$

whenever n is divisible by b, where $a \ge 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}), & \text{if } a > 1 \\ O(\log n), & \text{if } a = 1 \end{cases}$$

§8.3 Divide-and-Conquer Algorithms and Recurrence relations (3)

O符号定义：

假设 f 和 g 是从整数集合或实数集合到实数集合的函数，如果有常数 C 和 k，使得只要 x > k，就有 $|f(x)| \le C|g(x)|$，则f (x)是O(g (x))的。

Furthermore, when $n = b^k$, where k is a positive integer, $\mathbf{f(n) = C_1 n^{\log_b a} + C_2}$,

where $C_1 = f(1) + c/(a-1)$ and

$C_2 = -c/(a-1)$.

§ 8.3 Divide-and-Conquer Algorithms and Recurrence relations (4)

Theorem2 (Master Theorem)：

Let f be increasing function that satisfies the recurrence relation $f(n) = af(n/b)+cn^d$, whenever $n=b^k$, where k is a positive integer, $a \ge 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n)\ is \begin{cases} O(n^d) & if\ \ a < b^d \\ O(n^d \log n) & if\ \ a = b^d \\ O(n^{\log_b a}) & if\ \ a > b^d \end{cases}$$

§ 8.4 Generating Functions (1)

8.4.1 The concept of generating functions

（1） Ordinary generating function

Definition:

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

§8.4 Generating Functions (2)

Example 1：给出序列1，1，1，1，1的普通生成函数

$$f(x) = 1 + x + x^2 + x^3 + x^4 = \sum_{k=0}^{4} x^k = \frac{x^5 - 1}{x - 1}$$

Example 2：给出序列 $\left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n}\right)$ 的普通生成函数

$$f(x) = \binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{k=0}^{n}\binom{n}{k}x^k = (1+x)^n$$

§8.4 Generating Functions (3)

Example 3：给出序列 $\left(\binom{n-1}{0},-\binom{n}{1},\binom{n+1}{2},\cdots,(-1)^k\binom{n+k-1}{k},\cdots\right)$ 的普通生成函数

$$f(x)=\binom{n-1}{0}-\binom{n}{1}x^1+\binom{n+1}{2}x^2+\cdots+(-1)^k\binom{n+k-1}{k}x^k+\cdots$$

$$=\sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k=(1+x)^{-n}$$

§ 8.4 Generating Functions (4)

8.4.1 The concept of generating functions

(2) Exponential generating function

Definition:

The exponential generating function for the sequence $a_0,a_1,\ldots,a_k,\ldots$ of real numbers is the infinite series

$$f_e(x)=a_0+a_1\frac{x^1}{1!}+a_2\frac{x^2}{2!}+\cdots+a_n\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}a_n\frac{x^n}{n!}$$

§8.4 Generating Functions (5)

Example 4：给出序列$(P(n,0),P(n,1),\ldots,P(n,n))$的指数生成函数

$$f_e(x)=P(n,0)+P(n,1)\frac{x^1}{1!}+P(n,2)\frac{x^2}{2!}+\cdots+P(n,n)\frac{x^n}{n!}+0+\cdots$$

$$=\binom{n}{0}+\binom{n}{1}x^1+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n=(1+x)^n$$

§8.4 Generating Functions (6)

Example 5：给出序列 $(1, a, a^2, \cdots, a^n, \cdots)$ 的指数生成函数

$$f_e(x) = a_0 + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + \cdots + a_n \frac{x^n}{n!} + \cdots$$

$$= 1 + a \frac{x^1}{1!} + a^2 \frac{x^2}{2!} + \cdots + a^n \frac{x^n}{n!} + \cdots$$

$$= \sum_{k=0}^{\infty} a^k \frac{x^k}{k!} = e^{ax}$$

§ 8.4 Generating Functions (1)

8.4.2 Useful facts about power series

(1) The foundational operation of generating function

Theorem 1: Let $f(x)=\sum_{k=0}^{\infty} a_k x^k$ and $g(x)=\sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x)+g(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k \text{ and}$$

$$f(x)g(x)=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$$

8.4.2 Useful facts about power series

（2） The extended binomial theorem

Definition:

Let α be a real number and let k be a nonnegative integer. Then the extended binomial coefficient $C(\alpha,k)$ is defined by

$$\binom{\alpha}{k} = \begin{cases} \alpha(\alpha-1)\cdots(\alpha-k+1)/k! & \text{if } k>0 \\ 1 & \text{if } k=0 \end{cases}$$

§ 8.4 Generating Functions (3)

8.4.2 Useful facts about power series

(2) The extended binomial theorem

Example 1

$$\binom{-2}{3} = (-2)(-2-1)(-2-2)/3! = (-2)(-3)(-4)/3! = -4$$

$$\binom{1/2}{3} = (1/2)(1/2-1)(1/2-2)/3! = (1/2)(-1/2)(-3/2)/6 = 1/16$$

8.4.2 Useful facts about power series

(2) The extended binomial theorem

The extended binomial theorem:

Let x be a real number with $|x| < 1$ and let α be a real number. Then

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

8.4.2 Useful facts about power series

（2） The extended binomial theorem

Example 2 对于|x|<1 的任意 x，证明:

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty}(-1)^k \binom{n+k-1}{k} x^k$$

当n=1时 $$(1+x)^{-1} = \frac{1}{(1+x)} = \sum_{k=0}^{\infty}(-1)^k x^k$$

用-x替代x $$(1\text{-}x)^{-1} = \frac{1}{(1\text{-}x)} = \sum_{k=0}^{\infty} x^k$$

§ 8.4 Generating Functions (6)

8.4.2 Useful facts about power series

Example 3 证明 $(1-4x)^{-1/2}$ 是下列序列的普通生成函数。

$$\left(\binom{0}{0},\binom{2}{1},\binom{4}{2},\cdots,\binom{2n}{n},\cdots\right)$$

§8.4 Generating Functions (1)

8.4.3 Counting problems and generating functions

(1) 从 n 个不同物体中选取 r 个物体，其方法数为 $(1+x)^n$ 的幂级数展开式中 x^r 的系数 $C(n,r)$ 。

(2) 从 n 个不同物体中可重复选取 r 个物体，其方法数为 $(1+x+x^2+x^3+\ldots)^n$ 的幂级数展开式中 x^r 的系数。

§ 8.4 Generating Functions (2)

8.4.3 Counting problems and generating functions

例： 从 n 个不同物体中可重复选取 r 个物体的方法数为： $\binom{n+r-1}{r}$

$$f(x) = (1 + x + x^2 + \cdots)^n = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

8.4.3 Counting problems and generating functions

例： 从 n 个不同物体中可重复选取 r 个物体，每个物体至少选一次的方法数为：$\binom{r-1}{n-1}$

$$f(x) = (x + x^2 + \cdots)^n = \sum_{r=0}^{\infty} \binom{r-1}{n-1} x^r$$

8.4.3 Counting problems and generating functions

例：从 n 个不同物体中选取 r 个物体的方法数为：$\binom{n}{r}$

$$f(x)=(1+x)^n=\sum_{r=0}^{\infty}\binom{n}{r}x^r=\sum_{r=0}^{\infty}P(n,r)\frac{x^r}{r!}$$

例： 求 1,3,5,7,9 五个数字组成 r 位数的个数。其中7,9 出现的次数为偶数，其它数字出现的次数不加限制。

§ 8.4 Generating Functions (1)

8.4.4 Using generating functions to solve recurrence relations

Example 1:用生成函数求解下列递推关系式

$$\begin{cases} a_n = 3a_{n-1} & (n \geq 1) \\ a_0 = 2 \end{cases}$$

特征根 r=3

所以递推关系的解形式为：$a_n = b3^n$

由初始条件解为： $a_n = 2 \times 3^n$

§8.4 Generating Functions (2)

利用生成函数求解递推关系的步骤:

（1）用 f(x)表示序列 $(a_0,a_1,a_2,\ldots)$ 的普通生成函数。即

$$f(x)=\sum_{n=0}^{\infty}a_nx^n$$

（2）利用递推关系将生成函数化为关于 f(x)的方程。

（3）求解出 f(x)。

（4）将f(x)表达式展开成幂级数形式，x^n 的系数就是递推关系的解。

8.4.4 Using generating functions to solve recurrence relations

Example 2:用生成函数求解下列递推关系

$$\begin{cases} a_n = a_{n-1} + 2(n-1)\ (n \geq 2) \\ a_1 = 2 \end{cases}$$

8.4.4 Using generating functions to solve recurrence relations

Example 3:用生成函数求解下列递推关系

$$\begin{cases} a_n = na_{n-1} + (-1)^n & n \geq 2 \\ a_0 = 1, a_1 = 0 \end{cases}$$

$$\begin{cases} a_n = na_{n-1} + (-1)^n & n \geq 2 \\ a_0 = 1, a_1 = 0 \end{cases}$$

设 $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ 下面序列的指数生成函数

$$(a_0, a_1, a_2, \cdots, a_n, \cdots)$$

$$f(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} [na_{n-1} + (-1)^n] \frac{x^n}{n!} = 1 + \sum_{n=2}^{\infty} na_{n-1} \frac{x^n}{n!} + \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$= 1 + x\sum_{n=2}^{\infty} a_{n-1} \frac{x^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n!} + (1-x) - (1-x)$$

$$= 1 + x\sum_{n=1}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} - (1-x)$$

$$= 1 + x\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} - x + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} - (1-x)$$

$$= x\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = xf(x) + e^{-x}$$

所以 $f(x) = \dfrac{e^{-x}}{1-x}$

$$f(x) = \frac{e^{-x}}{1-x} = \frac{1}{1-x} e^{-x} = (\sum_{n=0}^{\infty} x^n)(\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}) = \sum_{n=0}^{\infty} n! \left[\sum_{k=0}^{n} (-1)^k \frac{1}{k!}\right] \frac{x^n}{n!}$$

§8.4 Generating Functions (1)

8.4.5 Proving identities via generating functions

Example 1:设 n 为任意正整数，证明:

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

§ 8.5 Inclusion-exclusion (1)

8.5.1 The principle of inclusion-exclusion

The principle of inclusion-exclusion:

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$
\begin{aligned}
&|A_1 \cup A_2 \cup \cdots \cup A_n| \\
&= \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots \\
&\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|
\end{aligned}
$$

§8.5 Inclusion-exclusion (2)

8.5.1 The principle of inclusion-exclusion

Example 1：某班有学生60名，其中24名学生选修 A 课程，28名学生选修 B 课程，26名学生选修 C 课程，10名学生既选修 A 又选修 B，8名学生既选修 A 又选修 C，14名学生既选修 B 又选修 C，6名学生3门课程都选修了。问有多少学生对这3门课程都没有选修?

§ 8.6 Applications of inclusion-exclusion

（1） An alternative form of inclusion-exclusion

$$N(P_1', P_2', \ldots, P_n') = |S| - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

$$= N - \sum_{i=1}^{n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i, P_j) - \sum_{1 \le i < j < k \le n} N(P_i, P_j, P_k)$$

$$+ \cdots + (-1)^n N(P_1, P_2, \cdots P_n)$$

§8.6 Applications of inclusion-exclusion

（2）Counting the number of primes

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~				

（3）The number of onto functions

§ 8.6 Applications of inclusion-exclusion

Theorem:

Let m and n be positive integers with $m \geq n$. Then, there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1} C(n,n-1)1^m$$

onto functions from a set with m elements to a set with n elements.

§ 8.6 Applications of inclusion-exclusion

（4） Derangements

Theorem：

The number of derangements of a set with n elements is

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

§ 8.6 Applications of inclusion-exclusion

（4） Derangements

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} + \cdots$$

$$e^{-1} = \frac{D_n}{n!} + (-1)^{n+1} \frac{1}{(n+1)!} + (-1)^{n+2} \frac{1}{(n+2)!} \cdots$$

$$\left| e^{-1} - \frac{D_n}{n!} \right| < \frac{1}{(n+1)!} \qquad \lim_{n\to\infty} \frac{D_n}{n!} = e^{-1}$$

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

$$D_1 = 0, D_2 = 1$$