# 13-14 学年第二学期高等数学试题(A)参考答案

一、填空题（每题 4 分，共 20 分）

1 、 4　　2、$2x+2y-3z=0$　　3、$\frac{1}{2}\left(1-\mathrm{e}^{-4}\right)$　　4 、 $x^5 f_{uu}+2x^3 f_{uv}+x f_{vv}$

5、$2\left(\mathrm{e}^a-1\right)+\frac{\pi}{4}a\mathrm{e}^a$

二、选择题（每题 4 分，共 20 分）

6-10、ABCDD

三、（12分）解：因为$l=\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\frac{n+1}{n+2}=1$,所以收敛半径$R=1$,

收敛区间是$\left(-1,1\right)$.设$\sum\limits_{n=0}^{\infty}\frac{x^n}{n+1}=S(x)$,显然$S(0)=1$，于是$xS(x)=\sum\limits_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\left(|x|<1\right)$

两边求导得$\left[xS(x)\right]'=\sum\limits_{n=0}^{\infty}\left(\frac{x^{n+1}}{n+1}\right)'=\sum\limits_{n=0}^{\infty}x^n=\frac{1}{1-x}$

上式从0到$x$积分得，

$xS(x)=\int_0^x\frac{1}{1-t}\mathrm{d}t=-\ln(1-x).$

于是当$x\neq 0$时，有$S(x)=-\frac{1}{x}\ln(1-x)$，从而

$$S(x)=\begin{cases}-\frac{1}{x}\ln(1-x),0<|x|<1,\\ 1,x=0.\end{cases}$$

四、（共12分，每小题6分）1、解：（1）设过直线$L$且垂直于平面$\pi$的平面为$\pi_1$,

$L$的方向向量为$\vec{S}=(1,1,-1)\times(1,-1,1)=(0,-2,-2)$,

$\pi_1$的法向量$\vec{n}_1=\vec{S}\times\vec{n}=(0,-2,-2)\times(1,1,1)=(0,-2,2)$

在$L$上取点$\left(0,0,-1\right)$，则平面$\pi_1$的方程$-2(y-0)+2(z+1)=0$,即$y-z-1=0$

平面$\pi$与$\pi_1$的交线即为$L_0:\begin{cases}x+y+z=0\\ y-z-1=0\end{cases}$.

(2)在$L_0$上取点$(-1,1,0)$，$L_0$的方向向量$\vec{S}_0=(1,1,1)\times(0,1,-1)=(-2,1,1)$,

直线$L_0$的对称式$\frac{x+1}{-2}=\frac{y-1}{1}=\frac{z}{1}$,

参数式$\begin{cases}x=-1-2t\\ y=1+t\\ z=t\end{cases}$　绕$z$轴旋转所得的旋转曲面方程为

$\begin{cases}x^2+y^2=\left(-1-2t\right)^2+\left(1+t\right)^2\\ z=t\end{cases}$ 消去$t$，得$x^2+y^2-5z^2-6z-2=0.$

2、解：$\lim\limits_{t\to0}\frac{1}{\pi t^2}\iiint\limits_{x^2+y^2+z^2\le t^2} f(\sqrt{x^2+y^2+z^2})\mathrm{d}x\mathrm{d}y\mathrm{d}z.=\lim\limits_{t\to0}\frac{1}{\pi t^2}\left[\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^t f\left(r\right)r^2\mathrm{d}r\right]$

$$=\lim_{t\to0}\frac{2\pi\cdot2\cdot\int_0^t f(r)r^2\mathrm{d}r}{\pi t^2}=\lim_{t\to0}\frac{4\pi f(t)t^2}{2\pi t}=\lim_{t\to0}2f(t)t=0$$

五、（10分，每题5分）

1、解：设$F(x)=\int_0^x f(t)\mathrm{d}t$,则$F'(x)=f(x)$.故

$$\int_0^1\mathrm{d}x\int_x^1\mathrm{d}y\int_x^y f(x)f(y)f(z)\mathrm{d}z=\int_0^1 f(x)\mathrm{d}x\int_x^1 f(y)\mathrm{d}y\int_x^y f(z)\mathrm{d}z$$

$$=\int_0^1 f(x)\mathrm{d}x\int_x^1 f(y)F(z)\Big|_x^y\mathrm{d}y=\int_0^1 f(x)\mathrm{d}x\int_x^1 F(y)-F(x)\mathrm{d}F(y)$$

$$=\int_0^1 f(x)\cdot\left[\frac{1}{2}F^2(y)-F(x)F(y)\right]\Big|_x^1\mathrm{d}x$$

$$=\int_0^1 f(x)\cdot\left[\frac{1}{2}F^2(1)-F(x)F(1)-\left(\frac{1}{2}F^2(x)-F^2(x)\right)\right]\mathrm{d}x$$

$$=\int_0^1\left[\frac{1}{2}F^2(1)-F(x)F(1)-\left(\frac{1}{2}F^2(x)-F^2(x)\right)\right]\mathrm{d}F(x)$$

$$=\frac{1}{2}F^2(1)F(x)-\frac{1}{2}F^2(x)F(1)+\frac{1}{2}\cdot\frac{1}{3}F^3(x)\Big|_0^1$$

$$=\left(\frac{1}{2}F^3(1)-\frac{1}{2}F^3(1)+\frac{1}{3!}F^3(1)\right)-\left(\frac{1}{2}F^2(1)F(0)-\frac{1}{2}F^2(0)F(1)+\frac{1}{3!}F^3(0)\right)$$

$$=\frac{1}{3!}F^3(1)-\left[\frac{1}{2}F^2(1)F(0)-\frac{1}{2}F^2(0)F(1)+\frac{1}{3!}F^3(0)\right]$$

$F(1)=\int_0^1 f(t)\mathrm{d}t=6,F(0)=\int_0^0 f(t)\mathrm{d}t=0,$故

$$\int_0^1\mathrm{d}x\int_x^1\mathrm{d}y\int_x^y f(x)f(y)f(z)\mathrm{d}z=\frac{1}{3!}F^3(1)\text{=}36.$$

2、解：设空间区域的体积为$V$,由高斯公式知$V\text{=}\frac{1}{3}\oiint\limits_{\Sigma}x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y$，

因为在球面Σ上,$\sqrt{x^2+y^2+z^2}\text{=}a,$

根据上式有$I=a\oiint\limits_{\Sigma}(x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y)=3aV=3a\cdot\frac{4\pi}{3}a^3=4\pi a^4.$

六、（10分)解：$P=\dfrac{-y}{4x^2+y^2},Q=\dfrac{x}{4x^2+y^2}$,

则$\dfrac{\partial P}{\partial y}=\dfrac{y^2-4x^2}{\left(4x^2+y^2\right)^2}=\dfrac{\partial Q}{\partial x},(x,y)\neq(0,0)$

作足够小的椭圆$C$:$\begin{cases}x=\dfrac{\delta}{2}\cos\theta \\ y=\delta\sin\theta\end{cases}$$\left(\theta\in[0,2\pi],C\text{取逆时针方向}\right)$，于是由格林公式有

$\oint_{L+C^-}\dfrac{x\mathrm{d}y-y\mathrm{d}x}{4x^2+y^2}=0$,即得$\oint_L\dfrac{x\mathrm{d}y-y\mathrm{d}x}{4x^2+y^2}=\oint_C\dfrac{x\mathrm{d}y-y\mathrm{d}x}{4x^2+y^2}=\int_0^{2\pi}\dfrac{\frac{1}{2}\delta^2}{\delta^2}\mathrm{d}\theta=\pi.$

七、(10)解：由题知 $u=x-2y,v=x+3y$. 所以 $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial x}=1,\quad \dfrac{\partial u}{\partial y}=-2,\quad \dfrac{\partial v}{\partial y}=3.$

由链导法则得

$$\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\&=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}.\end{aligned}$$

所以

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v\partial u}\frac{\partial u}{\partial x}$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2},$$

同理得

$$\frac{\partial z}{\partial y}=-2\frac{\partial z}{\partial u}+3\frac{\partial z}{\partial v},$$

$$\frac{\partial^2 z}{\partial x\partial y}=-2\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial u\partial v}+3\frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial^2 z}{\partial y^2}=4\frac{\partial^2 z}{\partial u^2}-12\frac{\partial^2 z}{\partial u\partial v}+9\frac{\partial^2 z}{\partial v^2}.$$

把以上各式带入题中方程得

$$5\frac{\partial^2 z}{\partial u\partial v}=\frac{\partial z}{\partial v}.$$

八、（6）解　$L: x^2+y^2=1$ 正向，$P=-\left(x^2+y^2\right)\dfrac{\partial f}{\partial y}, Q=\left(x^2+y^2\right)\dfrac{\partial f}{\partial x}$

$$\oint_L P\mathrm{d}x+Q\mathrm{d}y=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}\sigma=2\iint_D\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y+\iint_D\left(x^2+y^2\right)\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\mathrm{d}y$$

另一方面

$$\oint_L P\mathrm{d}x+Q\mathrm{d}y=\oint_L\left(x^2+y^2\right)\left(-\frac{\partial f}{\partial y}\mathrm{d}x+\frac{\partial f}{\partial x}\mathrm{d}y\right)=\oint_L-\frac{\partial f}{\partial y}\mathrm{d}x+\frac{\partial f}{\partial x}\mathrm{d}y$$

$$=\iint_D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\mathrm{d}y$$　　其中，圆域$D=\left\{(x，y)\mid x^2+y^2\le 1\right\}$

以上两式相减：$\displaystyle\iint_{x^2+y^2\le 1}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y$

$$=\frac{1}{2}\left[\iint_{x^2+y^2\le 1}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}\sigma-\iint_{x^2+y^2\le 1}\left(x^2+y^2\right)\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}\sigma\right]$$

$$=\frac{1}{2}\iint_{x^2+y^2\le 1}\mathrm{e}^{-\left(x^2+y^2\right)}\mathrm{d}\sigma-\frac{1}{2}\iint_{x^2+y^2\le 1}\left(x^2+y^2\right)\mathrm{e}^{-\left(x^2+y^2\right)}\mathrm{d}\sigma$$

$$=\frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\mathrm{e}^{-r^2}\mathrm{d}r-\frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^3\mathrm{e}^{-r^2}\mathrm{d}r=\frac{\pi}{2\mathrm{e}}$$